Some new results on the coefficients of cyclotomic and of inverse cyclotomic polynomials

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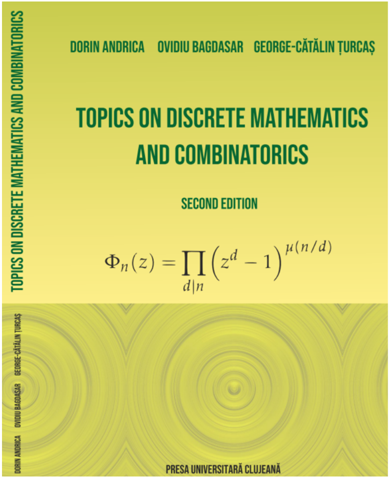
Key source of information



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Overview

1. Cyclotomic polynomials
   - Integral formula
   - Recursive formula
2. Inverse cyclotomic polynomials
   - Integral formula
   - Recursive formula
3. Binary and ternary polynomials

Joint work with *Ovidiu Bagdasar* (University of Derby, UK)

Presentation mainly based on the following resources

Andrica, D., and Bagdasar, O., *On cyclotomic polynomial coefficients*, Malays. J. Math. Sci., in: Proceedings of "Groups, Group Rings, and Related Topics - 2017" (GGRRT 2017), 19–22 Nov 2017, Khorfakan, UAE **14**(3) (2020), 389–402.

Andrica, D., and Bagdasar, O., *Some remarks on a general family of complex polynomials*, Appl. Anal. Discr. Math. **13** (2019), no. 1, 605–618.

Andrica, D., and Bagdasar, O., *Some remarks on the coefficients of cyclotomic polynomials*. In: J. Guàrdia, N. Minculete, D. Savin, M. Vela, A. Zekhnini (eds.), New Frontiers in Number Theory and Applications, Trends in Mathematics, Birkhäuser, Cham, 29–49, 2024.

Andrica, D., Bagdasar, O., Țurcaș, G.-C., *Topics on Discrete Mathematics and Combinatorics*, Cluj University Press, 2nd Edition, 2024.

Andrica, D., Bagdasar, O., and Țurcaș, G.-C., *An integral formula for the coefficients of the inverse cyclotomic polynomial*, An. Șt. Univ. Ovidius Constanța Ser. Mat. **33**(1) (2025), 33–48.

The Cauchy integral formula (1)

The Cauchy integral formula is a key result in complex analysis and it is also an important tool in analytic combinatorics.

A function $h : G \to \mathbb{C}$ is called *meromorphic at* z_0 if $h(z)$ can be written as a quotient of two analytic functions f, g in a neighbourhood $\mathcal{U}$ of z_0:

$$\forall z \in \mathcal{U} \setminus \{z_0\} \, : \, h(z) = \frac{f(z)}{g(z)}.$$

In this case, for all $z \neq z_0$ in a disk centered at z_0, we also have

$$h(z) = \sum_{n \geq -m} c_n (z - z_0)^n$$

If m is the largest number for which $c_{-m} \neq 0$ in this series expansion, then z_0 is called a *pole of order* m.

The coefficient c_{-1} of $(z - z_0)^{-1}$ in this series expansion is called *the residue of* h *at the point* z_0, also denoted as $\operatorname{Res}(h, z_0)$.

The Cauchy integral formula: (2) Residue Theorem

The important Residue theorem due to Cauchy relates global properties of a meromorphic function, its integral along closed curves, to purely local characteristics at designated points, the residues at poles.

Theorem (Cauchy's residue theorem)

Let $h(z)$ be a meromorphic function in the region Ω and let Γ be a simple loop in Ω along which the function is analytic. Then

$$\frac{1}{2\pi i}\int_{\Gamma} h(z)\,\mathrm{d}z = \sum_{s} Res(h(z), z = s),$$

where the sum is extended to all poles of s of $h(z)$ enclosed by Γ.

The Cauchy integral formula (3)

Theorem (Cauchy's integral formula)

Let $f(z) = \sum_{j\geq 0} c_j z^j$ *be analytic in a disk centered at* 0, *and let* Γ *be a curve in the interior of this disk, which winds around exactly once (in positive orientation), then we have*

$$c_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z^{j+1}} \, \mathrm{d}z.$$

Theorem (Cauchy's multi-variable integral formula)

Let $F(z_1, \ldots, z_k) = \frac{U(z_1,\ldots,z_k)}{V(z_1,\ldots,z_k)} = \sum_{j_1,\ldots,j_k \in \mathbb{Z}} c_{j_1,\ldots,j_k} z_1^{j_1} \cdots z_k^{j_k}$, *be a rational function. If* $V \neq 0$ *at the origin of* $\mathbb{R}^k$, *then the coefficients*

$$c_{j_1,\ldots,j_k} = \frac{1}{(2\pi i)^k} \int_T \frac{F(z_1, \ldots, z_k)}{z_1 \cdots z_k} z_1^{-j_1} \cdots z_k^{-j_k} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_k,$$

with T *a product of small circles around the coordinate axes of* $\mathbb{R}^k$.

The cyclotomic polynomial

The n-th cyclotomic polynomial Φ_n is defined by

$$\Phi_n(z) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} (z-\zeta_n^k) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} z^j, \tag{1}$$

- $\zeta_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ the first primitive nth root of the unity.
- degree of Φ_n: $\varphi(n)$, where φ denotes Euler's totient function.

First six cyclotomic polynomials:

$$\Phi_1(z) = z-1,\ \Phi_2(z) = z+1,\ \Phi_3(z) = z^2+z+1,$$
$$\Phi_4(z) = z^2+1,\ \Phi_5(z) = z^4+z^3+z^2+z+1,\ \Phi_6(z) = z^2-z+1.$$

It is well-known that every cyclotomic polynomial

- has integer coefficients
- is irreducible over $\mathbb{Z}$ ([31, Theorem 1, p.195]
- Φ_n is palindromic and monic

Applications of cyclotomic polynomials

- Solution of the problem of which regular n-gons are constructible with straightedge and compass (Gauss–Wantzel theorem).
- Elementary proofs of the existence of infinitely many prime numbers equal to 1 or -1 modulo n (special case of P.A.P).
- Witt's proof of Wedderburn's little theorem that every finite domain is a field.
- The "cyclotomic criterion" in the study of primitive divisors of Lucas and Lehmer sequences.
- Lattice-based cryptography.

Coefficients of cyclotomic polynomials

- Polynomials up to $n < 105$ only have 0, 1 and -1 as coefficients.

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- Mignotti 1883:
  - -2 first appears as the coefficient of z^7 of Φ_{105}
  - 2 first appears as a coefficient in Φ_{165}.
  - $\Phi_n(z)$ only has coefficients 0 and ± 1, if $n = pq$ with $p \neq q$ primes.
  - All coefficients of $\Phi_n(z)$ do not exceed 2 in absolute value for $n < 385$.

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- Bang 1895: For $n = pqr$ with $p < q < r$ odd primes, no coefficient of Φ_n is larger than $p - 1$.
- Schur 1931: The coefficients of cyclotomic polynomials can be arbitrarily large in absolute value.

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Theorem (Suzuki 1987 [48])

Every integer number can be a coefficient of a cyclotomic polynomial of a certain degree.

The history of these early results can be found in [35]. More details are given in [27], [28], [32], and to the monograph [17].

Algebraic form of $\Phi_n(z)$

Writing the polynomial $\Phi_n(z)$ in algebraic form, we obtain:

$$\Phi_n(z) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} z^j,$$

where $c_j^{(n)}$, $j = 0, 1, \ldots, \varphi(n)$, are the coefficients of $\Phi_n(z)$.

In order to get a unitary formula for $c_j^{(n)}$, we introduce the function

$$\Lambda_n(t) = \prod_{\substack{1 \le k \le n-1 \\ \gcd(k,n)=1}} \sin\left(t - \frac{k\pi}{n}\right). \tag{2}$$

For $n = 1, 2$ one obtains the following expressions:

$$\Lambda_1(t) = \sin(t - \pi) = -\sin t,$$
$$\Lambda_2(t) = \sin\left(t - \frac{\pi}{2}\right) = -\cos t.$$

The function $\Lambda_n(t)$

$$\Lambda_n(t) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \sin\left(t - \frac{k\pi}{n}\right). \tag{3}$$

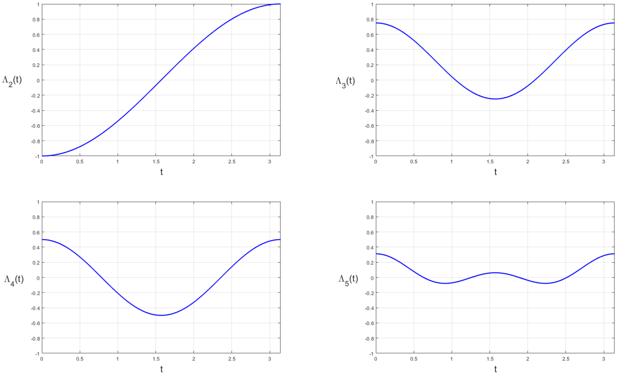

Figure: $\Lambda_n(t)$ $(0 \le t \le \pi)$ for (a) $n = 2$; (b) $n = 3$; (c) $n = 4$; (d) $n = 5$.

Integral formula for the coefficients of $\Phi_n(z)$

To prove the main result we use the following identity.

Lemma

Let $n \geq 3$ be a positive integer. The following formula holds:

$$\sum_{\substack{1 \leq k \leq n-1 \\ \gcd(k,n)=1}} k = \frac{n}{2}\varphi(n). \tag{4}$$

Theorem (Andrica & Bagdasar, MJMS 2020, Theorem 2.1)

The coefficients $c_j^{(n)}$ are given by the following integral formula:

$$c_j^{(n)} = \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \cdot \cos(\varphi(n) - 2j)t \, \mathrm{d}t, \quad j = 0, 1, \ldots, \varphi(n). \tag{5}$$

Reciprocity of coefficients of $\Phi_n(z)$

The coefficients of the cyclotomic polynomial are known to be reciprocal. Here we give an elegant proof based on the integral formula (5).

Theorem (Andrica & Bagdasar, MJMS 2020, Theorem 2.2)

The cyclotomic polynomial $\Phi_n(z)$ is reciprocal, that is its coefficients satisfy the following symmetry relations

$$c_j^{(n)} = c_{\varphi(n)-j}^{(n)}, \quad j = 0, 1, \ldots, \varphi(n).$$

Proof. Using formula (5), for every $j = 0, 1, \ldots, \varphi(n)$, we have

$$\begin{aligned} c_{\varphi(n)-j}^{(n)} &= \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \cos\left(\varphi(n) - 2(\varphi(n) - j)\right) t \,\mathrm{d}t \\ &= \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \cos\left(2j - \varphi(n)\right) t \,\mathrm{d}t \\ &= \frac{2^{\varphi(n)}}{\pi} \int_0^\pi \Lambda_n(t) \cdot \cos\left(\varphi(n) - 2j\right) t \,\mathrm{d}t = c_j^{(n)}. \end{aligned}$$

Direct/Alternate sum of coefficients. Mid-term of $\Phi_n(z)$

Theorem (Andrica & Bagdasar, MJMS 2020, Theorem 3.1)

Let $n \geq 3$. Expression $\Phi_n(1)$ has the following integral formula:

$$\Phi_n(1) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} = \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \cdot \frac{\sin(\varphi(n)+1)t}{\sin t} \, \mathrm{d}t. \tag{6}$$

Theorem (Andrica & Bagdasar, MJMS 2020, Theorem 3.2)

Let $n \geq 3$. Terms $\Phi_n(-1)$ have the following integral formula:

$$\Phi_n(-1) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} (-1)^j = \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \cdot \frac{\cos(\varphi(n)+1)t}{\cos t} \, \mathrm{d}t. \tag{7}$$

Mid-terms integral formula

$$m_n := c^{(n)}_{\frac{\varphi(n)}{2}} = \frac{2^{\varphi(n)}}{\pi} \int_0^{\pi} \Lambda_n(t) \, \mathrm{d}t. \tag{8}$$

The inverse cyclotomic polynomials

For every $n \in \mathbb{N}$, the inverse cyclotic polynomials $\Psi_n(z)$ are defined by

$$\Psi_n(z) = \prod_{\substack{1 \leqslant j \leqslant n \\ (j,n)>1}} \left(z - \zeta_n^j\right) = \frac{z^n - 1}{\Phi_n(z)} = \sum_{k=0}^{n-\varphi(n)} d_k^{(n)} z^k \tag{9}$$

1. $\Psi_n(z)$ has degree $n - \varphi(n)$, with integer coefficients.
2. If n is a prime, then $\Psi_n(z) = z - 1$.
3. Ψ_n is antipalindromic and monic.

The first inverse cyclotomic polynomials (n composite) are given by

$$\Psi_1(z) = 1,\ \Psi_4(z) = z^2 - 1,\ \Psi_6(z) = z^4 + z^3 - z - 1,\ \Psi_8(z) = z^4 - 1,$$
$$\Psi_9(z) = z^3 - 1,\ \Psi_{10}(z) = z^6 + z^5 - z - 1,\ \Psi_{12}(z) = z^8 + z^6 - z^2 - 1$$

Function $\Gamma_n(t)$

In order to get a unitary formula for $d_j^{(n)}$, we introduce the function

$$\Gamma_n(t) = \prod_{\substack{1 \le k \le n \\ \gcd(k,n)>1}} \sin\left(t - \frac{k\pi}{n}\right). \tag{10}$$

The first non-trivial values are $n = 4$ and $n = 6$ where we have

$$\begin{aligned}
\Gamma_4(t) &= \sin\left(t - \frac{2\pi}{4}\right)\sin\left(t - \frac{4\pi}{4}\right) \\
&= \sin\left(t - \frac{\pi}{2}\right)\sin\left(t - \pi\right) = \sin t \cdot \cos t = \frac{1}{2}\sin 2t. \\
\Gamma_6(t) &= \sin\left(t - \frac{2\pi}{6}\right)\sin\left(t - \frac{3\pi}{6}\right)\sin\left(t - \frac{4\pi}{6}\right)\sin\left(t - \frac{6\pi}{6}\right) \\
&= \sin\left(t - \frac{\pi}{3}\right)\sin\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{2\pi}{3}\right)\sin\left(t - \pi\right) \\
&= \frac{1}{8}\left(\sin 2t + \sin 4t\right).
\end{aligned}$$

Integral formula for the coefficients of $\Psi_n(z)$

To prove the coefficient formula we use the identity

$$\sum_{\substack{1\le k\le n \\ \gcd(k,n)>1}} k = \frac{n}{2}\left(n+1-\varphi(n)\right), \quad n \ge 3. \tag{11}$$

Theorem

The coefficients $d_j^{(n)}$, $j = 0, 1, \ldots, n-\varphi(n)$, of the inverse cyclotomic polynomial $\Psi_n(z)$ are given by the following integral formula:

$$d_j^{(n)} = (-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^{\pi}\Gamma_n(t)\cdot\sin\left(n-\varphi(n)-2j\right)t\,\mathrm{d}t. \tag{12}$$

Antireciprocity of coefficients of $\Psi_n(z)$

The coefficients of the inverse cyclotomic polynomial are antisymmetric.

Theorem (Ψ_n is antipalindromic)

The inverse cyclotomic polynomial $\Psi_n(z)$ is antipalindromic, that is its coefficients satisfy the following antisymmetry relation

$$d_j^{(n)} = -d_{n-\varphi(n)-j}^{(n)}, \quad j = 0, 1, \ldots, n - \varphi(n). \tag{13}$$

Proof. Using formula (12), for every $j = 0, 1, \ldots, \varphi(n)$, we have

$$\begin{aligned}
d_{n-\varphi(n)-j}^{(n)} &= (-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^\pi \Gamma_n(t)\sin\left(n-\varphi(n)-2(n-\varphi(n)-j)\right)t\,\mathrm{d}t \\
&= (-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^\pi \Gamma_n(t)\sin\left(2j+\varphi(n)-n\right)t\,\mathrm{d}t \\
&= -(-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^\pi \Gamma_n(t)\sin\left(n-\varphi(n)-2j\right)t\,\mathrm{d}t \\
&= -d_j^{(n)}.
\end{aligned}$$

Explicit formulas of coefficients of Φ_n and Ψ_n

Theorem (Cyclotomic polynomials, Endo 1974 [26])

The following formula holds

$$c_m^{(n)} = \sum_{i_1+2i_2+\cdots+mi_m=m} (-1)^{i_1+\cdots+i_m} \binom{\mu(n)}{i_1} \binom{\mu(n/2)}{i_2} \cdots \binom{\mu(n/m)}{i_m}, \quad (14)$$

where $(i_1, \ldots, i_m)$ runs over all the non-negative integral solutions of the equation $i_1 + 2i_2 + \cdots + mi_m = m$, for m a positive integer.

Theorem (Inverse cyclotomic polynomials)

For every $m, n \in \mathbb{N}$, $n \geq 1$, the following formula holds

$$d_m^{(n)} = \sum_{i_1+2i_2+\ldots+mi_m=m} (-1)^{i_1+\cdots+i_m+1} \binom{-\mu(n)}{i_1} \binom{-\mu(n/2)}{i_2} \cdots \binom{-\mu(n/m)}{i_m} \quad (15)$$

Recursive formulas

The following recursive formulae for the coefficients of Φ_n and Ψ_n are proved in [10], [11, Section 8.5], and [9], respectively

Theorem (Cyclotomic polynomials)

The following relation holds for every $k = 2, \dots, \varphi(n)$*:*

$$c_k^{(n)} = -\frac{1}{k}\left[\rho(n,k) + \rho(n,k-1)c_1^{(n)} + \cdots + \rho(n,1)c_{k-1}^{(n)}\right]. \tag{16}$$

Theorem (Inverse cyclotomic polynomials)

The following relation holds for every $k = 2, \dots, n - \varphi(n)$*:*

$$d_k^{(n)} = \frac{1}{k}\left[-\rho(n,k) + \rho(n,k-1)d_1^{(n)} + \cdots + \rho(n,1)d_{k-1}^{n}\right]. \tag{17}$$

Here $\rho(n, j)$ denote the Ramanujan sums.

Ramanujan Sums

The **Möbius function** μ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k, \\ 0 & \text{if } n = p^2 m, \end{cases}$$

where p is prime and $p_1, \ldots, p_k$ are distinct prime numbers.

For positive integers n and j, the **Ramanujan sum** is defined as

$$\rho(n, j) = \sum_{\gcd(a,n)=1} e^{2\pi i \frac{a}{n} j},$$

where the sum is over all integers a with $1 \leq a \leq n$ and $\gcd(a, n) = 1$.

The following formula due to Hölder

$$\rho(n, j) = \frac{\mu\left(\frac{n}{\gcd(n,j)}\right)\varphi(n)}{\varphi\left(\frac{n}{\gcd(n,j)}\right)}$$

often appears under the name of **Von Sterneck's function**.

Consequences

The previous formulas can be simplified as

Corollary (Cyclotomic polynomials)

The coefficients of Φ_n satisfy the following relation:

$$c_k^{(n)} = -\frac{\varphi(n)}{k}\left[\frac{\mu\left(\frac{n}{\gcd(n,k)}\right)}{\varphi\left(\frac{n}{\gcd(n,k)}\right)} + \frac{\mu\left(\frac{n}{\gcd(n,k-1)}\right)}{\varphi\left(\frac{n}{\gcd(n,k-1)}\right)}c_1^{(n)} + \cdots + \frac{\mu\left(\frac{n}{\gcd(n,1)}\right)}{\varphi\left(\frac{n}{\gcd(n,1)}\right)}c_{k-1}^{(n)}\right]. \tag{18}$$

Corollary (Inverse cyclotomic polynomials)

The coefficients of Ψ_n satisfy the following relation:

$$d_k^{(n)} = \frac{\varphi(n)}{k}\left[-\frac{\mu\left(\frac{n}{\gcd(n,k)}\right)}{\varphi\left(\frac{n}{\gcd(n,k)}\right)} + \frac{\mu\left(\frac{n}{\gcd(n,k-1)}\right)}{\varphi\left(\frac{n}{\gcd(n,k-1)}\right)}d_1^{(n)} + \cdots + \frac{\mu\left(\frac{n}{\gcd(n,1)}\right)}{\varphi\left(\frac{n}{\gcd(n,1)}\right)}d_{k-1}^{(n)}\right] \tag{19}$$

Results for binary polynomials Φ_n and Ψ_n

Suppose $n = pq$, where $p < q$ are distinct primes. In this case, the polynomials Φ_n are said to be **binary** and has the following expression

$$\Phi_{pq}(z) = \frac{(z^{pq} - 1)(z - 1)}{(z^p - 1)(z^q - 1)}.$$

Let u be the unique integer such that $0 < u < p$ and $uq \equiv -1 \pmod{p}$. The number of positive coefficients of $\Phi_{pq}(z)$ is $\frac{(p-u)(uq+1)}{p}$ [24].

Theorem (Lam 1996 [34])

For $n = pq$, where $p < q$ are primes, write r, s for the unique positive integers such that $\varphi(n) = (p-1)(q-1) = pr + qs$. Then the coefficients of Φ_n are given by the following formulas

$$c_k^{(n)} = \begin{cases} 1, \text{ if and only if } k = ip + jq \text{ with } i \in [0, r], j \in [0, s]; \\ -1, \text{ if and only if } k = ip + jq - pq, \ i \in [r+q, q-1], j \in [s+1, p-1] \\ 0, \text{ otherwise.} \end{cases}$$

In this case $d_k^{(n)} = -1$ for all $1 \le k \le p-1$, and that $d_p^{(n)} = d_{p+1}^{(n)} = 0$. The inverse cyclotomic polynomial has the following simple form

$$\Psi_n(z) = \frac{(z^p - 1)(z^q - 1)}{z - 1},$$

which is in turn equal to

$$\Psi_n(z) = z^{p+q-1} + \cdots + z^q - z^{p-1} - \cdots - z^2 - z - 1.$$

Hence, all the coefficients of these polynomials belong to the set $\{-1, 0, 1\}$, that is, the binary inverse cyclotomic polynomials are **flat**.

Results for ternary polynomials Φ_n and Ψ_n

Suppose $n = pqr$, where $p < q < r$ are primes.

The ternary polynomial Φ_n has the following form

$$\Phi_{pqr}(z) = \frac{(1 - z^{pqr})(1 - z^r)(1 - z^q)(1 - z^p)}{(1 - z^{qr})(1 - z^{pr})(1 - z^{pq})(1 - z)}.$$

Proposition (Coefficients of ternary cyclotomic polynomials)

1. $c_k^{(n)} = 1$, *for all* $1 \leq k \leq p - 1$;
2. $c_p^{(n)} = c_{p+1}^{(n)} = 0$.

Proposition (Results for Ramanujan triples)

For every $n = pqr$ *where* (p, q, r) *is a Ramanujan triple, we have*

1. $c_k^{(n)} = 0$ *for every* $p < k < q$;
2. $c_k^{(n)} = -1$ *for every* $q \leq k < r$;
3. $c_r^{(n)} = -2$.

Definition

We call a triple of natural numbers (p, q, r) a **Ramanujan triple** if p, q, r are primes and $p < q < r < 2p$.

In his landmark 1919 paper [45], Ramanujan presented a novel proof of Bertrand's postulate, together with a generalization.

An important consequence is that for all $x \geq 11$ one has

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq 3,$$

where π represents the prime-counting function. By considering arbitrarily large values of x, there exist infinitely many distinct Ramanujan triples.

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By Ramanujan's result, there is an infinite family of $n = pqr$ for which $c_r^{(n)} = -2$. While the first Ramanujan triple is $(7, 11, 13)$, where $n = 1001$, this polynomial has $c_{13}^{(1001)} = -2$ and $c_{199}^{(1001)} = -2$. but the polynomial has no coefficients of absolute value strictly greater than 2. However, for the Ramanujan triple $(17, 19, 29)$ and $n = 9367$ there are coefficients of larger absolute value such as $c_{3107}^{(9367)} = -4$.

Results for ternary polynomials Ψ_n

Suppose $n = pqr$, where $p < q < r$ are primes.

The ternary polynomial Ψ_n has the following form

$$\Psi_n(z) = \frac{(z^{pq}-1)(z^{qr}-1)(z^{rp}-1)(z-1)}{(z^p-1)(z^q-1)(z^r-1)}.$$

Proposition (Coefficients of ternary inverse cyclotomic polynomials)

1. *$d_1^{(n)} = 1$ and $d_k^{(n)} = 0$, for all $2 \le k \le p-1$;*
2. *$d_p^{(n)} = -1$ and $d_{p+1}^{(n)} = 1$.*

Proposition (Results for Ramanujan triples)

For every $n = pqr$ where (p, q, r) is a Ramanujan triple, we have

1. *$d_k^{(n)} = 0$ for all $p+2 \le k \le q-1$ and $d_q^{(n)} = -1$;*
2. *$d_{q+1}^{(n)} = 1$ and $d_k^{(n)} = 0$ for all $q+2 \le k < r$;*
3. *$d_r^{(n)} = -1$.*

We note that, combined with Ramanujan's result on the infinitude of such triples, these propositions give precise values for the first r terms in an infinite family of inverse cyclotomic polynomials.

One would be mislead to think that for $n = pqr$, where (p, q, r) is a Ramanujan triple, the inverse cyclotomic polynomials are **flat**. However, this is **NOT** true.

Indeed, computations with Magma confirmed that

- for $n = 11 \cdot 13 \cdot 19$ we have $d_{53}^{(n)} = 2$;
- for the Ramanujan triple $(101, 103, 109)$ where $n = 1133927$ we obtain $d_{15651}^{(n)} = -16$.

Some computational remarks

Arnold and Monagan [1] presented three optimized algorithms for computing the coefficients of the nth cyclotomic polynomial:

- Polynomial divisions optimized with FFT.
- Quotient of sparse power series, improved by treating $\Phi_n(X)$ as a truncated power series.
- "Big prime algorithm" that generates terms sequentially to reduce memory cost.

They produced extensive data on $\Phi_n(X)$ for $n = pqr < 10^8$.

With the recurrence relations above, and their analogues for the cyclotomic polynomials, one can easily go into much higher range for special cases of ternary cyclotomic polynomials. For instance, when

$$n = (10^8 + 7) \cdot (10^8 + 37) \cdot (10^8 + 39) = 1000000830000197500010101$$

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