Nonlinear Aspects in Dynamic Geometry

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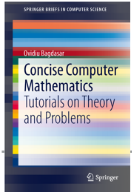

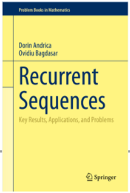

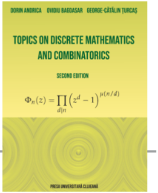

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Overview

1. Dynamic geometry in the triangle
    - Definitions and notations
    - Some illustrative classic examples
2. Kasner triangles with real and complex parameter
    - General formula of the *n*th iteration
    - Orbit classification: convergent/divergent/periodic/dense
3. Kasner iterations with sequences of parameters
4. Nested cevian triangles
    - A general convergence result
    - Iterations defined by the power curve of a triangle
5. Generalized Kasner iterations: nonlinearity and chaotic behaviour

Joint with *D. Andrica* (Babeș-Bolyai University, Cluj Napoca, Romania)

Some published works on the topic

[1] D. Andrica, D. Şt. Marinescu, *Dynamic Geometry Generated by the Circumcircle Midarc Triangle*. In: Rassias, Th. M. and Pardalos, P. M. (eds.) Analysis, Geometry, Nonlinear Optimization and Applications, World Scientific Publishing Company Ltd, Singapore, 2022.

[2] D. Andrica, O. Bagdasar, D. Şt. Marinescu, *Dynamic geometry of Kasner triangles with a fixed weight*, Int. J. Geom. **11**(2) (2022), 101–110.

[3] D. Andrica, O. Bagdasar, *On the dynamic geometry of Kasner triangles with complex parameter*. In: Proc. of 7th IACMC 2022, Zarqa University, Zarqa, Jordan, 11–13 May 2022. Nature Springer, Berlin, 2023.

[4] D. Andrica, O. Bagdasar, *Dynamic geometry of Kasner quadrilaterals with complex parameter*. Mathematics, **10** (2022), Article 3334, 12 pp.

[5] D. Andrica, O. Bagdasar, D. Şt. Marinescu, *New results on the dynamic geometry generated by sequences of nested triangles*, Carpath. J. Math. **39**(3) (2023).

[6] D. Andrica, O. Bagdasar, *Dynamic geometry of Kasner polygons with complex parameter*. In: Proc. of 27th ICDEA 2022, Paris-Saclay University, Paris, France, 17–22 July 2022. Nature Springer, Berlin, 2024.

[7] D. Andrica, O. Bagdasar, *On Kasner Triangles Defined by a Sequence of Parameters*. In: Proc. of 28th ICDEA, Phitsanulok, Thailand, July 17-21, 2023, Springer Proc. in Mathematics & Statistics, Vol 485. Springer, Cham (2025).

Dynamic geometry

Given a fixed plane configuration $\mathcal{F}_0$ and a sequence of surjective plane transformations $(T_n)_{n\geq 0}$, one can define a *dynamic geometry* as the iterative process described by

$$\mathcal{F}_0 \xrightarrow{T_0} \mathcal{F}_1 \xrightarrow{T_1} \mathcal{F}_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} \mathcal{F}_n \xrightarrow{T_n} \mathcal{F}_{n+1} \xrightarrow{T_{n+1}} \cdots .$$

After n steps the initial configuration $\mathcal{F}_0$ is transformed into $\mathcal{F}_n$ by the composition $T_{n-1} \circ \cdots \circ T_0$. The initial configuration $\mathcal{F}_0$ can be any general pattern defined using polygons, circles, or associated geometric elements (see [4], [6], [9], [12], [17]).

General questions

1. Compute the n-step configuration $\mathcal{F}_n$ and its geometric elements;
2. Investigate convergence properties of the sequence $(\mathcal{F}_n)_{n\geq 0}$;
3. If the above sequence is convergent, then what is the limit?;
4. Obtain properties of the initial configuration $\mathcal{F}_0$ from the study of the geometry of $\mathcal{F}_n$ for some $n \geq 1$.

Dynamic geometries in the triangle

Starting from $A_0B_0C_0$, one builds the sequence $(A_nB_nC_n)_{n\geq 0}$ recursively. Simple configurations inspired dynamic geometries generated by [6], [18]:

- incircle/circumcircle, bisector/pedal/orthic triangle
- hyperbolic, spherical nested triangles (Guo, 2024 [14]).

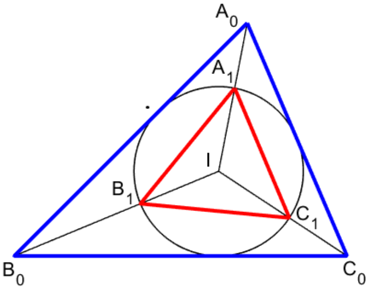


Figure 1: Iterations defined by the incircle I and bisector triangles.

Some contributors: Abbot [1], Braica & Marinescu [8], Chang & Davis [9], Clarke [10], J. Ding, Hitt & Zhang [12], [16], Ismailescu & Jacobs [17].

Some types of convergence

We consider the particular case when the configuration $\mathcal{F}_n$ is a triangle $A_nB_nC_n$, obtained iteratively from a given triangle $A_0B_0C_0$.

The sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is said to be:

- **convergent** if the sequences $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ of vertices are convergent. If the limits are A, B and C, respectively, then the limit of the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is the triangle ΔABC; If $A = B = C$, i.e., ΔABC is degenerated to a point, then we say that the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ **converges to a point**.
- **convergent in shape** if the sequences $(\widehat{A}_n)_{n\geq 0}$, $(\widehat{B}_n)_{n\geq 0}$, $(\widehat{C}_n)_{n\geq 0}$ are convergent, and the limits are not zero. In particular, when

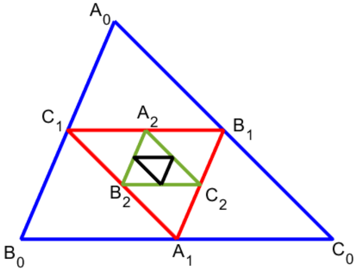
$$\lim_{n\to\infty} \widehat{A}_n = \lim_{n\to\infty} \widehat{B}_n = \lim_{n\to\infty} \widehat{C}_n = \frac{\pi}{3},$$

  $(\Delta A_nB_nC_n)_{n\geq 0}$ **converges in the shape of an equilateral triangle**.

Clearly, if the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges to a non-degenerated triangle ΔABC, then it is also convergent in shape of the triangle ΔABC.

Example 1. Median triangles

Points A_{n+1}, B_{n+1}, C_{n+1} are the mid-points of (B_nC_n), (C_nA_n), (A_nB_n).



Denoting the complex coordinates of A_n, B_n, C_n by a_n, b_n, c_n we get:

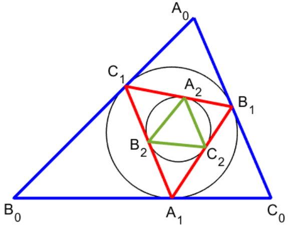
$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} g_0 \\ g_0 \\ g_0 \end{pmatrix}. \quad (1)$$

Convergence

- to the centroid: $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = G_0$.
- in shape: $\widehat{A}_n = \widehat{A}_0,\ \widehat{B}_n = \widehat{B}_0,\ \widehat{C}_n = \widehat{C}_0,\ n \geq 1$.

Example 2. Incircle-circumcircle triangles

The triangle $\Delta A_{n+1}B_{n+1}C_{n+1}$ is defined such that A_{n+1}, B_{n+1}, C_{n+1} are incircle intersections with (B_nC_n), (C_nA_n), (A_nB_n), respectively.



Note: $r_{n+1} \leq \frac{1}{2}r_n$, $n \geq 0$.

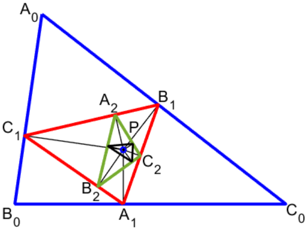
$$\begin{pmatrix} \widehat{A}_{n+1} \\ \widehat{B}_{n+1} \\ \widehat{C}_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \widehat{A}_n \\ \widehat{B}_n \\ \widehat{C}_n \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \widehat{A}_0 \\ \widehat{B}_0 \\ \widehat{C}_0 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{3} \\ \frac{\pi}{3} \\ \frac{\pi}{3} \end{pmatrix}. \quad (2)$$

Convergence

- To a single point called the Poncelet point of $\Delta A_0B_0C_0$.
- in shape to equilateral: $\lim_{n\to\infty} \widehat{A}_n = \lim_{n\to\infty} \widehat{B}_n = \lim_{n\to\infty} \widehat{C}_n = \frac{\pi}{3}$.

Example 3. Pedal triangle of a point P

Let P be an arbitrary point and $A_0B_0C_0$ a triangle.
Points A_{n+1}, B_{n+1}, C_{n+1} the feet of the perpendiculars from the point P on the sides of the triangle (B_nC_n), (C_nA_n), (A_nB_n).



Theorem 1 (Neuberg)

The triangles $A_0B_0C_0$ and $A_3B_3C_3$ are similar.

Convergence in shape (see [11], [22]). This implies that the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ is periodic in shape, having period 3.

Example 4^* **Orthic triangles.**

Sequences of nested triangles and Kasner triangles

Let $\mathcal{T}_0 = \Delta A_0 B_0 C_0$ be a triangle in the Euclidean plane.

A sequence of triangles $(\mathcal{T}_n)_{n\geq 0}$ is **nested** if the vertices A_{n+1}, B_{n+1}, C_{n+1} are on the segments $(B_n C_n)$, $(C_n A_n)$, $(A_n B_n)$, respectively.

A sequence of triangles $(\mathcal{T}_n)_{n\geq 0}$ is **Kasner** if the vertices A_{n+1}, B_{n+1}, C_{n+1} are on the support straight lines of $B_n C_n$, $C_n A_n$, $A_n B_n$, respectively.

Classical Kasner triangles: $\frac{A_{n+1}C_n}{B_n C_n} = \frac{B_{n+1}A_n}{C_n A_n} = \frac{C_{n+1}B_n}{A_n B_n} = \alpha, \quad n \geq 0.$

- For $\alpha \in [0,1]$, the Kasner triangles are nested.
- For $\alpha = \frac{1}{2}$ we obtain the sequence of median triangles.

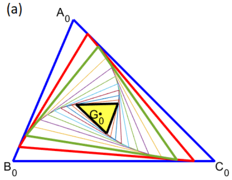


Figure 2: Kasner triangles $(A_n B_n C_n)_{n=0}^{10}$. (a) $\alpha = 0.1$; (b) $\alpha = 0.025$.

Basic results for Kasner triangles (real parameter)

Let a_n, b_n, c_n denote the complex coordinates of the vertices A_n, B_n, C_n, defined recursively for $n \geq 0$ as:

$$\begin{cases} a_{n+1} = \alpha b_n + (1-\alpha) c_n \\ b_{n+1} = \alpha c_n + (1-\alpha) a_n \\ c_{n+1} = \alpha a_n + (1-\alpha) b_n. \end{cases} \tag{3}$$

Theorem 2 (Main Theorem [4])

1) The sequence $(A_n B_n C_n)_{n\geq 0}$ is convergent if and only if $\alpha \in (0,1)$. When the sequence is convergent, its limit is the degenerated triangle at $G_0(g_0)$, the centroid of $A_0 B_0 C_0$.
2) When $\alpha \in (0,1)$, the order of convergence to $g_0 = \frac{a_0+b_0+c_0}{3}$ of the sequences $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$ is $\left(\frac{1}{3\alpha^2-3\alpha+1}\right)^{n/2}$.

Remark. When α is complex, the triangles $A_n B_n C_n$ are not always nested.

Kasner triangles with complex parameter - Formulae (1)

The system (3) can be written in matrix form as

$$X_{n+1} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 1-\alpha \\ 1-\alpha & 0 & \alpha \\ \alpha & 1-\alpha & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = TX_n, \tag{4}$$

where $X_n = (a_n, b_n, c_n)^T$, $n \geq 0$. In this notation one can write

$$X_n = T^n X_0. \tag{5}$$

The characteristic polynomial of T is

$$p_T(u) = (u-1)\left(u^2 + u + 3\alpha^2 - 3\alpha + 1\right),$$

whose roots are $u_0 = 1$ and for $\omega = \exp\left(\frac{2\pi i}{3}\right)$ we have

$$u_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \alpha\sqrt{3}i = \omega - \alpha\sqrt{3}i, \tag{6}$$

$$u_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i + \alpha\sqrt{3}i = \omega^2 + \alpha\sqrt{3}i. \tag{7}$$

Kasner triangles with complex parameter - Formulae (2)

It follows that for every $n \geq 1$ we have

$$T = F^{-1}\,\mathrm{diag}[1, u_1, u_2]\,F, \quad T^n = F^{-1}\,\mathrm{diag}[1, u_1^n, u_2^n]\,F,$$

$$F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad F^{-1} = \frac{1}{3}\overline{F} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \tag{8}$$

By multiplication with $(a_0, b_0, c_0)^T$ one obtains

$$\begin{aligned}
a_n &= \frac{a_0 + b_0 + c_0}{3} + \frac{a_0 + b_0\omega^2 + c_0\omega}{3}u_1^n + \frac{a_0 + b_0\omega + c_0\omega^2}{3}u_2^n \\
b_n &= \frac{a_0 + b_0 + c_0}{3} + \frac{a_0\omega + b_0 + c_0\omega^2}{3}u_1^n + \frac{a_0\omega^2 + b_0 + c_0\omega}{3}u_2^n \\
c_n &= \frac{a_0 + b_0 + c_0}{3} + \frac{a_0\omega^2 + b_0\omega + c_0}{3}u_1^n + \frac{a_0\omega + b_0\omega^2 + c_0}{3}u_2^n,
\end{aligned} \tag{9}$$

that is $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$ are obtained by the Discrete Fourier Transform matrix F. If $a_0^2 + b_0^2 + c_0^2 - a_0b_0 - b_0c_0 - c_0a_0 \neq 0$ (i.e., $\Delta A_0B_0C_0$ is not equilateral), then the coefficients of u_1^n or u_2^n are not zero.

Dynamical properties: Notations (1)

By (6) and (7) we get

$$u_1 = \sqrt{3}i \left[\alpha - \left(\frac{1}{2} - \frac{\sqrt{3}}{6}i\right)\right], \tag{10}$$

$$u_2 = -\sqrt{3}i \left[\alpha - \left(\frac{1}{2} + \frac{\sqrt{3}}{6}i\right)\right], \tag{11}$$

which can also be written as

$$\begin{aligned} u_1 &= r_1 e^{2\pi i\theta_1} = \sqrt{3}i\,(\alpha - z_1) = \sqrt{3}\,(\alpha - z_1)\, e^{\frac{2\pi i}{4}}, \\ u_2 &= r_2 e^{2\pi i\theta_2} = -\sqrt{3}i\,(\alpha - z_2) = \sqrt{3}\,(\alpha - z_2)\, e^{\frac{6\pi i}{4}}, \end{aligned} \tag{12}$$

where $r_1, r_2, \theta_1, \theta_2$ are real numbers, where we denote

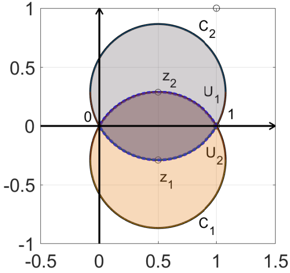
$$z_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}i, \quad z_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}i. \tag{13}$$

Dynamical properties: Notations (2)

$$D_1 = \left\{ z \in \mathbb{C} \,:\, |z - z_1| < \frac{\sqrt{3}}{3} \right\}, \qquad D_2 = \left\{ z \in \mathbb{C} \,:\, |z - z_2| < \frac{\sqrt{3}}{3} \right\}$$

$$C_1 = \left\{ z \in \mathbb{C} \,:\, |z - z_1| = \frac{\sqrt{3}}{3} \right\}, \qquad C_2 = \left\{ z \in \mathbb{C} \,:\, |z - z_2| = \frac{\sqrt{3}}{3} \right\}.$$

Notice that $z_2 - z_1 = \frac{\sqrt{3}}{3} i$, while $z_1 \in C_2$ and $z_2 \in C_1$. The circles C_1, C_2, the disks D_1, D_2 and the points z_1, z_2 are depicted in Figure 3.



Dynamical properties: Convergent orbits (1)

Theorem 3

1° *The sequence of triangles $(A_nB_nC_n)_{n\geq 0}$ is convergent if and only if $\alpha \in D_1 \cap D_2$.*

2° *When the sequence $(A_nB_nC_n)_{n\geq 0}$ is convergent, its limit is the degenerated triangle at G_0, the centroid of the initial triangle $A_0B_0C_0$.*

For $0 < \alpha < 1$ one has $\alpha \in D_1 \cap D_2$, when A_{n+1}, B_{n+1}, C_{n+1} are interior points of $[B_n, C_n]$, $[A_n, C_n]$ and $[A_n, B_n]$, as shown in Figure 4.

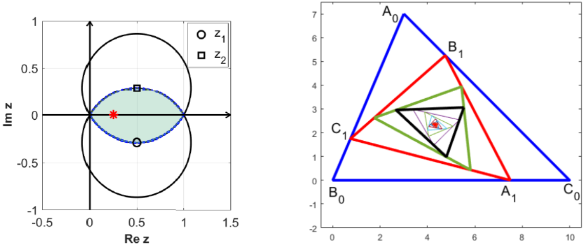


Figure 4: Convergent orbits (right) obtained for $\alpha = 0.25$ (left).

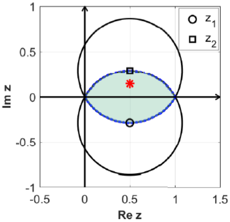
Dynamical properties: Convergent orbits (2)

Theorem 4

1° *The sequence of triangles* $(A_n B_n C_n)_{n\geq 0}$ *is convergent if and only if* $\alpha \in D_1 \cap D_2$.

2° *When the sequence* $(A_n B_n C_n)_{n\geq 0}$ *is convergent, its limit is the degenerated triangle at* G_0*, the centroid of the initial triangle* $A_0 B_0 C_0$.

On the other hand, when the parameter $\alpha \in D_1 \cap D_2$ is not real, the orbit is convergent, but the points are not aligned any more, as in Figure 5.



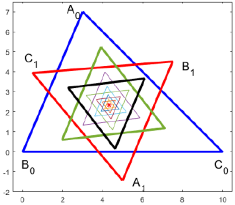


Figure 5: Convergent orbits (right) obtained for $\alpha = \frac{1}{2} + \frac{\sqrt{3}}{12} i$ (left).

Dynamical properties: Divergent orbits

If $\max\{r_1, r_2\} > 1$, then $\alpha \in \operatorname{int}(D_1 \cap D_2)^c$ by (12) either u_1^n or u_2^n are divergent. Therefore, by (9) the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ are divergent (as long as the corresponding coefficient is not vanishing, which is the case when the starting triangle a_0, b_0, c_0 are not the coordinates of an equilateral triangle.

Figure 6 depicts a divergent iteration.

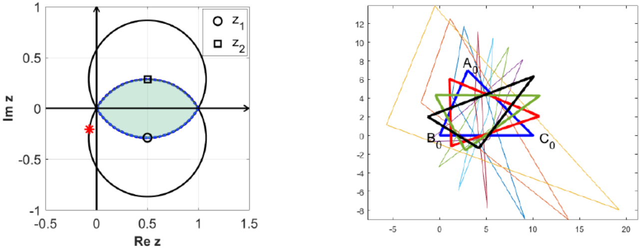


Figure 6: Divergent orbits (right) obtained for $\alpha = z_1 + \frac{\sqrt{3}}{3}(\cos 3 + i \sin 3)$ (left).

Dynamical properties: Periodic orbits

Periodic orbits are obtained for $r_1 = 0$ ($\alpha = z_1$), or $r_2 = 0$ ($\alpha = z_2$). The orbits obtained for $\alpha = z_1$ and $\alpha = z_2$ are depicted in Figure 7. Notice that these sequences have period 2 and $a_{n+2} = a_n$, for $n \geq 1$.

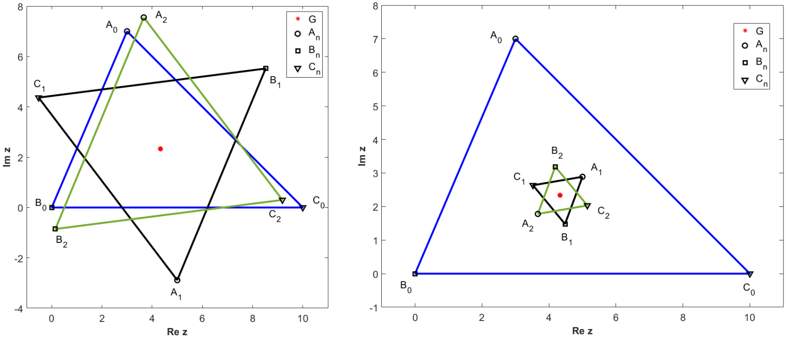


Figure 7: Periodic orbits: $\alpha = z_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}i$ (left) and $\alpha = z_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}i$ (right).

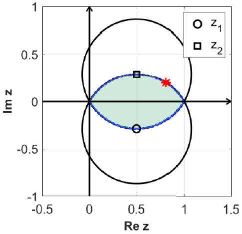
If $r_1 = r_2 = 1$, then $|\alpha - z_1| = |\alpha - z_2| = \frac{\sqrt{3}}{3}$, so $\alpha \in C_1 \cap C_2 = \{0, 1\}$, when the periodic sequence satisfy $a_{n+3} = c_{n+2} = b_{n+1} = a_n$ for $n \geq 0$.

Dynamical properties: Dense orbits

When $0 < \min\{r_1, r_2\} < \max\{r_1, r_2\} = 1$ but with θ_1 irrational, the orbits of $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ are dense within circles. First, assume that $r_1 = \max\{r_1, r_2\} = 1$, i.e., α is on the upper arc $C_1 \cap D_2$.

Theorem 5

If $\theta_1 \in \left[\frac{5\pi}{12}, \frac{13\pi}{12}\right]$ is irrational, then the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ are each dense within a circle centred at g_0.



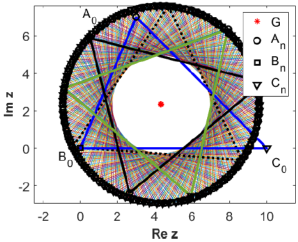


Figure 8: Orbits for $\theta_1 = p/k = 3/5$ where $\alpha = z_1 + \frac{\sqrt{3}}{3} e^{2\pi i\left(\frac{1}{4}+\frac{3}{5}\right)}$ (left).

Dynamical properties: Convergent subsequences

Theorem 6

If for $0 < p < k$ integers, $\theta_1 = \frac{p}{k} \in \left[\frac{5\pi}{12}, \frac{13\pi}{12}\right]$ is an irreducible fraction, then $u_1 = e^{2\pi i \frac{p}{k}}$ and by (9) the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ are such that for each $j = 0, \dots, k-1$ we get

$$\lim_{n\to\infty} a_{nk+j} = g_0 + \frac{a_0 + b_0\omega^2 + c_0\omega}{3} u_1^j$$

$$\lim_{n\to\infty} b_{nk+j} = g_0 + \frac{a_0\omega + b_0 + c_0\omega^2}{3} u_1^j$$

$$\lim_{n\to\infty} c_{nk+j} = g_0 + \frac{a_0\omega^2 + b_0\omega + c_0}{3} u_1^j. \tag{14}$$

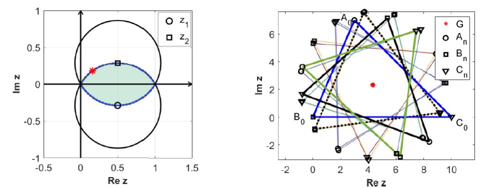


Figure 9: Orbits for $\theta_1 = p/k = 3/5$ where $\alpha = z_1 + \frac{\sqrt{3}}{3} e^{2\pi i\left(\frac{1}{4}+\frac{3}{5}\right)}$ (left).

Kasner triangles with a sequence of parameters

Let $(\alpha_n)_{n\geq 0}$ be a sequence of real or complex numbers.

The complex coordinates are now defined recursively for $n \geq 0$ as:

$$\begin{cases} a_{n+1} = \alpha_n b_n + (1-\alpha_n)c_n \\ b_{n+1} = \alpha_n c_n + (1-\alpha_n)a_n \\ c_{n+1} = \alpha_n a_n + (1-\alpha_n)b_n. \end{cases} \tag{15}$$

A natural problem is to characterise the sequences $(\alpha_n)_{n\geq 0}$ for which the sequence $(A_nB_nC_n)_{n\geq 0}$ is convergent (or dense, divergent, etc).

The system (15) can be written in matrix form as

$$X_{n+1} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \alpha_n & 1-\alpha_n \\ 1-\alpha_n & 0 & \alpha_n \\ \alpha_n & 1-\alpha_n & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = T_n X_n, \tag{16}$$

where $X_n = (a_n, b_n, c_n)^T$, $n \geq 0$. In this notation one can write

$$X_{n+1} = T_n T_{n-1} \cdots T_0 X_0, \tag{17}$$

where the matrices $(T_n)_{n\geq 0}$ are circulant and doubly stochastic.

Characteristic polynomials

As before, the characteristic polynomial of T_n is

$$p_{T_n}(u) = (u-1)\left(u^2 + u + 3\alpha_n^2 - 3\alpha_n + 1\right),$$

whose roots are 1 and denoting $\omega = \exp\left(\frac{2\pi i}{3}\right)$ we have

$$u_n = -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \alpha_n\sqrt{3}i = \omega - \alpha_n\sqrt{3}i, \tag{18}$$

$$v_n = -\frac{1}{2} - \frac{\sqrt{3}}{2}i + \alpha_n\sqrt{3}i = \omega^2 + \alpha_n\sqrt{3}i. \tag{19}$$

It follows that for every $n \geq 1$ we have

$$T_n = F^{-1}\,\mathrm{diag}[1, u_n, v_n]\,F,$$

where the matrices F and F^{-1}seen before, hence

$$T_{n-1}\cdots T_0 = F^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & \prod_{j=0}^{n-1} u_j & 0\\ 0 & 0 & \prod_{j=0}^{n-1} v_j\end{pmatrix} F.$$

Exact formulae for sequence terms (1)

Denoting

$$x_n = \prod_{j=0}^{n-1} u_j = \prod_{j=0}^{n-1} \left(\alpha_j \omega + (1-\alpha_j)\omega^2\right)$$

$$y_n = \prod_{j=0}^{n-1} v_j = \prod_{j=0}^{n-1} \left(\alpha_j \omega^2 + (1-\alpha_j)\omega\right),$$

the latest formula simplifies to

$$\begin{aligned} a_n &= g_0 + Mx_n + Ny_n \\ b_n &= g_0 + \omega M x_n + \omega^2 N y_n \\ c_n &= g_0 + \omega^2 M x_n + \omega N y_n, \quad n \geq 0, \end{aligned} \tag{20}$$

where

$$M = \frac{a_0 + b_0\omega^2 + c_0\omega}{3}, \quad N = \frac{a_0 + b_0\omega + c_0\omega^2}{3}. \tag{21}$$

A convergence result (real-valued sequences)

Theorem 7 (Real-valued sequences)

If $M \cdot N \neq 0$, then the sequence $(\Delta A_n B_n C_n)_{n\geq 0}$ is convergent to a triangle $A^* B^* C^*$ with the same centroid as $\Delta A_0 B_0 C_0$, if and only if the sequence $(x_n)_{n\geq 0}$ is convergent to x^*. The coordinates of the limit triangle are

$$\begin{aligned} a^* &= g_0 + Mx^* + N\,\overline{x^*} \\ b^* &= g_0 + \omega M x^* + \omega^2 N\,\overline{x^*} \\ c^* &= g_0 + \omega^2 M x^* + \omega N\,\overline{x^*}. \end{aligned} \tag{22}$$

We now give a sufficient condition for the convergence to the centroid G_0.

Corollary 8

If $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = g_0$, and the sequence of triangles $(A_n B_n C_n)_{n\geq 0}$ converges to the centroid G_0 of the initial triangle. This follows by (20).

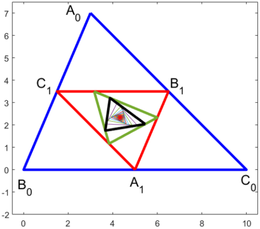
Kasner triangles with sequences of real parameters

Theorem 9 (Convergence Theorem.)

The sequence of perimeters $p_n = \alpha_n + \beta_n + \gamma_n$ *converges to zero if and only if* $\lim_{n\to\infty} \prod_{k=0}^{n} [1 - 3\alpha_k(1-\alpha_k)] = 0$.

Lemma 10

Let $(a_n)_{n\geq 0}$ *be a real sequence with* $a_n \in (0,1]$ *with* $\sum_{n=0}^{\infty} a_n$ *convergent. Then the product* $\prod_{n=0}^{\infty}(1-a_n)$ *converges to a positive limit.*



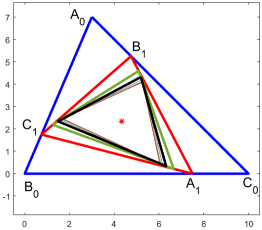


Figure 10: Kasner triangles $(A_nB_nC_n)_{n=1}^{10}$: (a) $\alpha_n = \frac{1}{n+1}$; (b) $\alpha_n = \frac{1}{2^{n+1}}$.

Kasner triangles with sequences of complex parameters

For a complex sequence $(\alpha_n)_{n\geq 0}$, the Kasner iterations may converge to

- a point: $\alpha_n = \frac{1}{2} + \frac{1}{n+4} i$;
- a triangle: $\alpha_n = \frac{1}{2^{n+1}} i$.

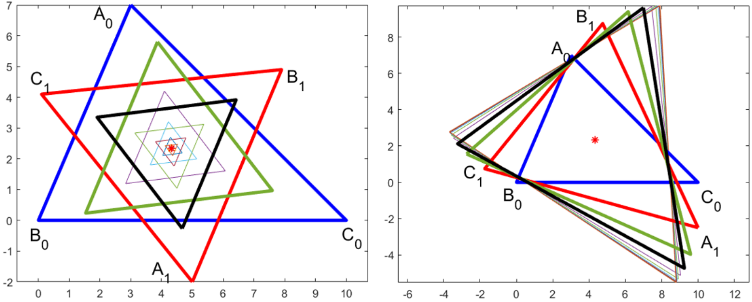


Figure 11: Sequence of Kasner triangles $A_n B_n C_n$ with $n = 0, \ldots, 10$, computed for (left) $\alpha_n = \frac{1}{2} + \frac{1}{n+4} i$; (right) $\alpha_n = \frac{1}{2^{n+1}} i$ (not nested!).

Generalized Kasner iterations

General Kasner iterations can be defined by signed ratios x_n, y_n, z_n in which the points A_{n+1}, B_{n+1}, C_{n+1} divide the segments $[B_nC_n]$, $[C_nA_n]$ and $[A_nB_n]$, depending on (a_n, b_n, c_n), or the sides $(\alpha_n, \beta_n, \gamma_n)$.

$$\begin{cases} a_{n+1} = x_n b_n + (1 - x_n)\, c_n \\ b_{n+1} = y_n c_n + (1 - y_n)\, a_n \\ c_{n+1} = z_n a_n + (1 - z_n)\, b_n, \quad n \geq 0, \end{cases} \tag{23}$$

which can be written in matrix form as

$$X_{n+1} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & x_n & 1 - x_n \\ 1 - y_n & 0 & y_n \\ z_n & 1 - z_n & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad n \geq 0. \tag{24}$$

- **Kasner iterations with a sequence:** $x_n = y_n = z_n = \alpha_n$, $n \geq 0$.
- **Nested Triangles:** Ratios satisfy $x_n, y_n, z_n \in [0, 1]$.
  - **Bisector triangles:** $x_n = \frac{\gamma_n}{\beta_n + \gamma_n}$, $y_n = \frac{\gamma_n}{\gamma_n + \alpha_n}$, $z_n = \frac{\beta_n}{\alpha_n + \beta_n}$.
  - **Symmedian triangles:** $x_n = \frac{\gamma_n^2}{\beta_n^2 + \gamma_n^2}$, $y_n = \frac{\gamma_n^2}{\gamma_n^2 + \alpha_n^2}$, $z_n = \frac{\beta_n^2}{\alpha_n^2 + \beta_n^2}$.

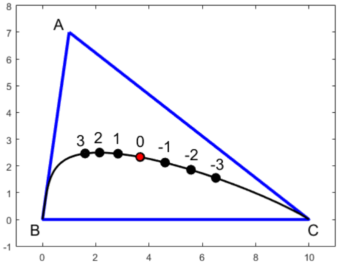
Nonlinear aspects: The power curve of a triangle

In a triangle ABC, denote by a, b, c the complex coordinates of the vertices and by α, β, γ the side lengths of the triangle. For a real number s we consider the point $Q^{(s)}$ having the complex coordinate

$$q^{(s)} = \frac{\alpha^s a + \beta^s b + \gamma^s c}{\alpha^s + \beta^s + \gamma^s}. \tag{25}$$

$Q^{(s)}$ has barycentric coordinates $\left(\frac{\alpha^s}{\alpha^s+\beta^s+\gamma^s}, \frac{\beta^s}{\alpha^s+\beta^s+\gamma^s}, \frac{\gamma^s}{\alpha^s+\beta^s+\gamma^s}\right)$.

Remarkable points on Γ: $Q^{(0)} = G$ (centroid), $Q^{(1)} = I$ (incentre), while $Q^{(2)} = K$ (symmedian, or Lemoine point) of ΔABC.



Nested triangles defined by points on the power curve

Let $(\Delta A_n B_n C_n)_{n\geq 0}$ be the sequence of triangles defined as: for each positive integer n and each real number s consider the point $Q_n^{(s)}$ situated on the power curve Γ_n of triangle $A_n B_n C_n$, with the complex coordinate

$$q_n^{(s)} = \frac{\alpha_n^s a_n + \beta_n^s b_n + \gamma_n^s c_n}{\alpha_n^s + \beta_n^s + \gamma_n^s}. \tag{26}$$

The points A_{n+1}, B_{n+1}, C_{n+1} are defined recursively by the intersections $(B_n C_n) \cap A_n Q_n^{(s)}$, $(C_n A_n) \cap B_n Q_n^{(s)}$, and $(A_n B_n) \cap C_n Q_n^{(s)}$, respectively.

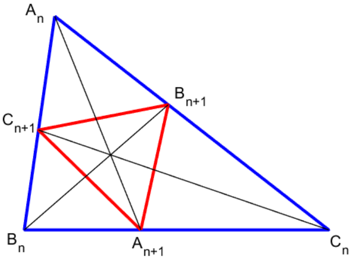


Figure 12: Bisector triangle iterations.

A general convergence result for nested triangles

Theorem 11 (Theorem 2.2, [5])

Let $(\Delta A_n B_n C_n)_{n\geq 0}$ *be a sequence of nested triangles, whose perimeters are denoted by* $p_n = \alpha_n + \beta_n + \gamma_n$*. Then* $(\Delta A_n B_n C_n)_{n\geq 0}$ *is convergent to a point if and only if the sequence of perimeters* $(p_n)_{n\geq 0}$ *converges to zero.*

Remark. The condition in Theorem 11 may not hold, even in the case of iterations defined by usual cevians. A striking example is that of the sequence of triangles defined by the feet of the altitudes of a triangle, proved to be chaotic (see [18]).

Result for selected points on the power curve.

Theorem 12 (Theorem 4.4, [5])

If $s \in [0, 2]$*, then the sequence* $(\Delta A_n B_n C_n)_{n\geq 0}$ *is convergent to a point.*

This result completes the work in [17], where it was proved that the sequence $(\mathcal{T}_n)_{n\geq 0}$ converges in the shape of an equilateral triangle.

Nonlinear iterations: Explicit Formulation

For a fixed real number s, the sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ in Theorem 12, satisfy the nonlinear recursive system:

$$\begin{cases} a_{n+1} = \frac{|a_n-b_n|^s}{|c_n-a_n|^s+|a_n-b_n|^s} b_n + \frac{|c_n-a_n|^s}{|c_n-a_n|^s+|a_n-b_n|^s} c_n \\ b_{n+1} = \frac{|b_n-c_n|^s}{|a_n-b_n|^s+|b_n-c_n|^s} c_n + \frac{|a_n-b_n|^s}{|a_n-b_n|^s+|b_n-c_n|^s} a_n \\ c_{n+1} = \frac{|c_n-a_n|^s}{|b_n-c_n|^s+|c_n-a_n|^s} a_n + \frac{|b_n-c_n|^s}{|b_n-c_n|^s+|c_n-a_n|^s} b_n, \quad n \geq 0. \end{cases} \tag{27}$$

The matrix form of the system is

$$X_{n+1} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\gamma_n^s}{\beta_n^s+\gamma_n^s} & \frac{\beta_n^s}{\beta_n^s+\gamma_n^s} \\ \frac{\gamma_n^s}{\gamma_n^s+\alpha_n^s} & 0 & \frac{\alpha_n^s}{\gamma_n^s+\alpha_n^s} \\ \frac{\beta_n^s}{\alpha_n^s+\beta_n^s} & \frac{\alpha_n^s}{\alpha_n^s+\beta_n^s} & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = T_n X_n, \quad n \geq 0,$$

where T_n is row-stochastic, $n \geq 0$, while α_n, β_n and γ_n are given by

$$\alpha_n = |b_n - c_n|, \quad \beta_n = |c_n - a_n|, \quad \gamma_n = |a_n - b_n|.$$

In this notation one can write

$$X_{n+1} = (T_n T_{n-1} \cdots T_1 T_0) X_0. \tag{28}$$

Numerical simulations: Case 1. $s = 0$

From the initial triangle $A_0(1+7i)$, $B_0(0)$ and $C_0(10)$ we compute the sequence $(\Delta A_n B_n C_n)_{n\geq 0}$ for different values of s.

For $s = 0$ the sequence $(\Delta A_n B_n C_n)_{n\geq 0}$ converges to the centroid G_0 of $A_0 B_0 C_0$. It is also convergent in the shape of the original triangle.

n	$\widehat{A_n}$	$\widehat{B_n}$	$\widehat{C_n}$	p_n	K_n	Q_n^s
0	60.2551	81.8699	37.8750	28.4728	35.0000	3.6667 + 2.3333i
1	60.2551	81.8699	37.8750	14.2364	8.7500	3.6667 + 2.3333i
2	60.2551	81.8699	37.8750	7.1182	2.1875	3.6667 + 2.3333i
3	60.2551	81.8699	37.8750	3.5591	0.5469	3.6667 + 2.3333i
4	60.2551	81.8699	37.8750	1.7796	0.1367	3.6667 + 2.3333i
5	60.2551	81.8699	37.8750	0.8898	0.0342	3.6667 + 2.3333i
6	60.2551	81.8699	37.8750	0.4449	0.0085	3.6667 + 2.3333i
7	60.2551	81.8699	37.8750	0.2224	0.0021	3.6667 + 2.3333i

Table 1: Angles (in degrees), perimeter and area of triangles $\Delta A_n B_n C_n$ calculated for $n = 0, \ldots, 9$, and sequence terms Q_n^s obtained for $s = 0$.

Numerical simulations: Case 2. $s = 1$

Starting from $A_0B_0C_0$, one builds the sequence $(A_nB_nC_n)_{n\geq 0}$ recursively, defined by the feet of the cevians through the point $Q_n^{(s)}$.

For $s = 1$ the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ converges to a point (Theorem 12), which is not currently known. It is also in the shape of an equilateral triangle, as proved by Jacobs and Ismailescu [17].

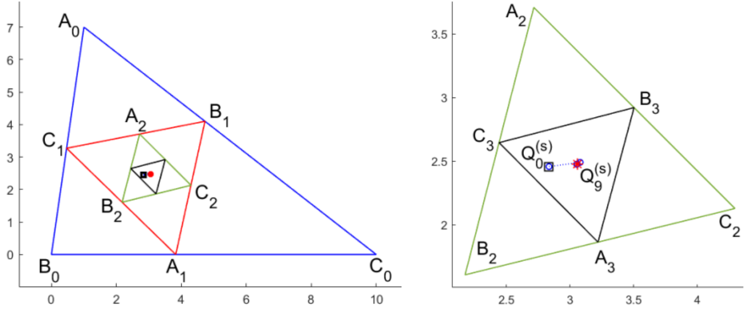


Figure 13: $\Delta A_nB_nC_n$, $n = 0, 1, 2, 3$, and $Q_n^{(s)}$, $n = 0, \dots, 9$, for $s = 1$.

Numerical simulations: Case 2. $s = 1$

For $s = 1$ the sequence $(\Delta A_n B_n C_n)_{n \geq 0}$ converges to a point (Theorem 12), which is not currently known. It is also in the shape of an equilateral triangle, as proved by Jacobs and Ismailescu [17].

n	$\widehat{A_n}$	$\widehat{B_n}$	$\widehat{C_n}$	p_n	K_n	Q_n^s
0	60.2551	81.8699	37.8750	28.4728	35.0000	2.8347 + 2.4585i
1	58.1565	66.5992	55.2442	13.2283	8.3620	3.0804 + 2.4916i
2	59.4254	61.7698	58.8047	6.5853	2.0855	3.0526 + 2.4760i
3	59.8490	60.4511	59.7000	3.2916	0.5213	3.0562 + 2.4784i
4	59.9618	60.1133	59.9249	1.6458	0.1303	3.0557 + 2.4781i
5	59.9904	60.0284	59.9812	0.8229	0.0326	3.0558 + 2.4781i
6	59.9976	60.0071	59.9953	0.4114	0.0081	3.0558 + 2.4781i
7	59.9994	60.0018	59.9988	0.2057	0.0020	3.0558 + 2.4781i

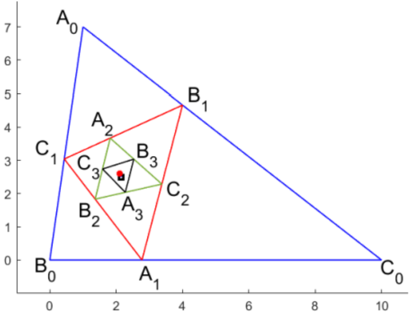
Table 2: Angles (in degrees), perimeter and area of triangles $\Delta A_n B_n C_n$ calculated for $n = 0, \ldots, 9$, and sequence terms Q_n^s obtained for $s = 1$.

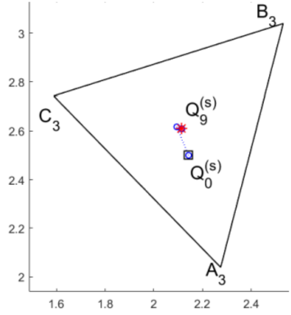
Numerical simulations: Case 3. $s = 2$

Starting from $A_0B_0C_0$, one builds the sequence $(A_nB_nC_n)_{n\geq 0}$ recursively, defined by the feet of the cevians through the point $Q_n^{(s)}$.

For $s = 2$ the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ convergences to a point, by Theorem 12. This limit point is not currently known.

The numerical simulations seem to indicate that the sequence of triangles converges in the shape of an equilateral triangle, but at this moment do not know how to prove this property.





Numerical simulations: Case 3. $s = 2$

Table 3 seems to indicate that the sequence of triangles converges in the shape of an equilateral triangle, but we do not know a proof of this.

n	$\widehat{A_n}$	$\widehat{B_n}$	$\widehat{C_n}$	p_n	K_n	Q_n^s
0	60.2551	81.8699	37.8750	28.4728	35.0000	2.1429 + 2.5000i
1	52.2670	50.8446	76.8885	12.5823	7.3269	2.0940 + 2.6171i
2	62.1740	63.4854	54.3406	6.0412	1.7468	2.1154 + 2.6088i
3	58.6416	58.1494	63.2090	3.0026	0.4331	2.1141 + 2.6095i
4	60.6049	60.8874	58.5077	1.4987	0.1080	2.1143 + 2.6094i
5	59.6801	59.5484	60.7715	0.7491	0.0270	2.1143 + 2.6094i
6	60.1555	60.2237	59.6209	0.3745	0.0067	2.1143 + 2.6094i
7	59.9212	59.8877	60.1912	0.1872	0.0017	2.1143 + 2.6094i
8	60.0391	60.0560	59.9048	0.0936	0.0004	2.1143 + 2.6094i
9	59.9804	59.9719	60.0477	0.0468	0.0001	2.1143 + 2.6094i

Table 3: Angles (in degrees), perimeter and area of triangles $\Delta A_n B_n C_n$ calculated for $n = 0, \ldots, 9$, and sequence terms Q_n^s obtained for $s = 2$.

Numerical simulations: Case 4. $s = 3$

Starting from $A_0B_0C_0$, one builds the sequence $(A_nB_nC_n)_{n\geq 0}$ recursively, defined by the feet of the cevians through the point $Q_n^{(s)}$.

For $s = 3$ the sequence $(\Delta A_nB_nC_n)_{n\geq 0}$ still seems to converge to a point. It does not seem to converge in the shape of an equilateral triangle. In fact, the limit in shape seems to be degenerate.

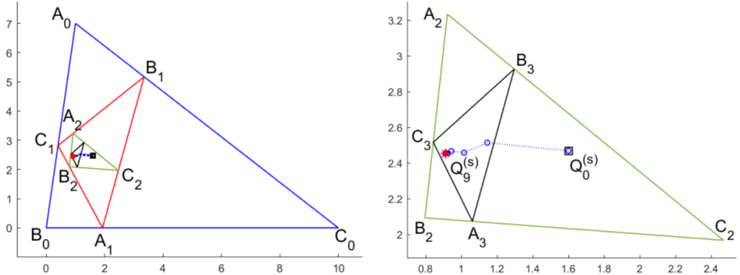


Figure 15: $\Delta A_nB_nC_n$, $n = 0, 1, 2, 3$, and $Q_n^{(s)}$, $n = 0, \ldots, 9$, for $s = 3$.

Numerical simulations: Case 4. $s = 3$

For $s = 3$ the sequence $(\Delta A_n B_n C_n)_{n \geq 0}$ still seems to converge to a point, but does not seem to converge in the shape of an equilateral triangle, as suggested by the numerical simulations in Table 4. In fact, the limit in shape seems to be degenerate.

n	$\widehat{A_n}$	$\widehat{B_n}$	$\widehat{C_n}$	p_n	K_n	Q_n^s
0	60.2551	81.8699	37.8750	28.4728	35.0000	$1.5994 + 2.4685i$
1	43.7770	36.0169	100.2061	12.3403	5.9474	$1.1448 + 2.5137i$
2	56.9816	88.0145	35.0038	4.8234	0.9616	$1.0149 + 2.4581i$
3	41.6992	32.5444	105.7564	1.9916	0.1456	$0.9430 + 2.4667i$
4	52.4537	94.4473	33.0990	0.7337	0.0213	$0.9248 + 2.4542i$
5	41.8269	30.7372	107.4359	0.2903	0.0030	$0.9147 + 2.4557i$
6	47.9812	100.6540	31.3648	0.1047	0.0004	$0.9122 + 2.4535i$
7	42.4043	29.7297	107.8660	0.0396	0.0001	$0.9109 + 2.4538i$

Table 4: Angles (in degrees), perimeter and area of triangles $\Delta A_n B_n C_n$ calculated for $n = 0, \ldots, 9$, and sequence terms Q_n^s obtained for $s = 3$.

Some Open Problems

We have proved that the sequence of nested triangles defined by points $Q^{(s)}$ on the power curve converge to a point, for $s \in [0,2]$.

In particular, this closes an open problem posed in 2006 by Ismailescu and Jacobs, recovered for the incentre $I = Q^{(1)}$.

Open problems:

Problem 1. *By Theorem 12, for every $s \in [0,2]$, the sequence $(\Delta A_n B_n C_n)_{n \geq 0}$ converges to a point X_s^*. Find this point!*

Problem 2. *For $s = 0$ the sequence $(\Delta A_n B_n C_n)_{n \geq 0}$ converges in the shape of the initial triangle, while for $s = 1$ the convergence is in the shape of an equilateral triangle. Find the real parameters s for which the convergence in shape holds, and characterize the shape.*

Problem 3. *Find the values $s \notin [0,2]$, for which the sequence $(\Delta A_n B_n C_n)_{n \geq 0}$ is convergent. Our numerical simulations seem to indicate that such values of s exist (e.g. $s = 3$), but other methods are needed.*

Orthic (pedal) triangle iterations

The triangle $\Delta A_{n+1}B_{n+1}C_{n+1}$ is defined by the feet of the altitudes from the vertices A_n, B_n, C_n on (B_nC_n), (C_nA_n), (A_nB_n), respectively.

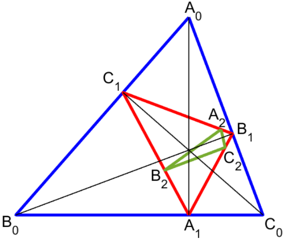


Figure 16: Pedal triangle iterations $\Delta A_nB_nC_n$, $n = 0, 1, 2, 3$.

- If all angles A_n, B_n, C_n, of $\Delta A_nB_nC_n$ are acute, then

$$A_{n+1} = \pi - 2A_n, \quad B_{n+1} = \pi - 2B_n, \quad C_{n+1} = \pi - 2C_n$$

- If the angle A_n is obtuse, then

$$A_{n+1} = 2A_n - \pi, \quad B_{n+1} = 2B_n, \quad C_{n+1} = 2C_n.$$

Chaotic case: Orthic triangles (1)

In the General Kasner iterations framework one can write

$$\begin{cases} a_{n+1} = x_n b_n + (1 - x_n)\, c_n \\ b_{n+1} = y_n c_n + (1 - y_n)\, a_n \\ c_{n+1} = z_n a_n + (1 - z_n)\, b_n, \quad n \geq 0, \end{cases} \tag{29}$$

where the ratios x_n, y_n, z_n are real numbers depending on a_n, b_n and c_n. Clearly, $x_n, y_n, z_n \in [0, 1]$ when $\Delta A_n B_n C_n$ is acute, but not when obtuse.

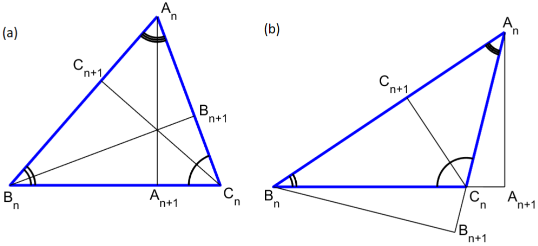


Figure 17: Pedal triangle of $\Delta A_n B_n C_n$ when this is (a) acute; (b) obtuse.

Chaotic case: Orthic triangles (2)

The signed ratios x_n, y_n and z_n can be written as

$$x_n = \frac{a_{n+1}-c_n}{b_n-c_n} = \frac{\beta_n}{\alpha_n}\cos C_n = \frac{1}{2} + \frac{\beta_n^2-\gamma_n^2}{\alpha_n^2} = \frac{1}{2} + \frac{|a_n-c_n|^2-|a_n-b_n|^2}{|b_n-c_n|^2}$$

$$y_n = \frac{b_{n+1}-a_n}{c_n-a_n} = \frac{\gamma_n}{\beta_n}\cos A_n = \frac{1}{2} + \frac{\gamma_n^2-\alpha_n^2}{\beta_n^2} = \frac{1}{2} + \frac{|b_n-a_n|^2-|b_n-c_n|^2}{|c_n-a_n|^2}$$

$$z_n = \frac{c_{n+1}-b_n}{a_n-b_n} = \frac{\alpha_n}{\gamma_n}\cos B_n = \frac{1}{2} + \frac{\alpha_n^2-\beta_n^2}{\gamma_n^2} = \frac{1}{2} + \frac{|c_n-b_n|^2-|c_n-a_n|^2}{|a_n-b_n|^2}.$$

- $\Delta A_n B_n C_n$ acute: $x_n, y_n, z_n \in [0,1]$ and $\Delta A_{n+1} B_{n+1} C_{n+1}$ is nested.
- $\Delta A_n B_n C_n$ right triangle: $\Delta A_{n+1} B_{n+1} C_{n+1}$ is degenerate.
- $\Delta A_n B_n C_n$ obtuse: $x_n, y_n, z_n \notin [0,1]$, $\Delta A_{n+1} B_{n+1} C_{n+1}$ is not nested.

The general system (29) can be written for all $n \geq 0$ as

$$\begin{cases} a_{n+1} = \left(\frac{1}{2} + \frac{|a_n-c_n|^2-|a_n-b_n|^2}{|b_n-c_n|^2}\right) b_n + \left(\frac{1}{2} - \frac{|a_n-c_n|^2-|a_n-b_n|^2}{|b_n-c_n|^2}\right) c_n \\ b_{n+1} = \left(\frac{1}{2} + \frac{|b_n-a_n|^2-|b_n-c_n|^2}{|c_n-a_n|^2}\right) c_n + \left(\frac{1}{2} - \frac{|b_n-a_n|^2-|b_n-c_n|^2}{|c_n-a_n|^2}\right) a_n \\ c_{n+1} = \left(\frac{1}{2} + \frac{|c_n-b_n|^2-|c_n-a_n|^2}{|a_n-b_n|^2}\right) a_n + \left(\frac{1}{2} - \frac{|c_n-b_n|^2-|c_n-a_n|^2}{|a_n-b_n|^2}\right) b_n. \end{cases} \tag{30}$$

Chaotic case: Orthic triangles (3)

Theorem 13 (Hobson's formula [18], [19])

If $\Delta A_n B_n C_n$, $n \geq 0$ are not degenerate, then the sides of the nth pedal triangle are given by the Hobson formula (correct within a sign):

$$\alpha_n = \pm\alpha_0 \cos A_0 \cos 2A_0 \cdots \cos 2^{n-1} A_0 = \pm\alpha_0 \cdot \frac{1}{2^n} \cdot \frac{\sin 2^n A_0}{\sin A_0}$$

$$\beta_n = \pm\beta_0 \cos B_0 \cos 2B_0 \cdots \cos 2^{n-1} B_0 = \pm\beta_0 \cdot \frac{1}{2^n} \cdot \frac{\sin 2^n B_0}{\sin B_0}$$

$$\gamma_n = \pm\beta_0 \cos C_0 \cos 2C_0 \cdots \cos 2^{n-1} C_0 = \pm\beta_0 \cdot \frac{1}{2^n} \cdot \frac{\sin 2^n C_0}{\sin C_0}.$$

By the Law of Sines in $\Delta A_0 B_0 C_0$ we get $\frac{\alpha_0}{\sin A_0} = \frac{\beta_0}{\sin B_0} = \frac{\gamma_0}{\sin C_0} = 2R_0$, so

$$\alpha_n = \pm\frac{R_0 \sin 2^n A_0}{2^{n-1}}, \quad \beta_n = \pm\frac{R_0 \sin 2^n B_0}{2^{n-1}}, \quad \gamma_n = \pm\frac{R_0 \sin 2^n C_0}{2^{n-1}}.$$

Convergence: Chaotic or cyclic iterations.

Here $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} p_n = 0$, but orthic iterations may not converge as in Theorem 2.2 [5], as they are not nested.

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