A Potpourri of Topics from Fractal Interpolation Theory

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Sept 09, 2025

Outline

- Iterated Function Systems and the Fractal Interpolation Problem
- Global and Local Fractal Interpolation
- Non-Stationary Fractal Interpolation
- Quaternionic Fractal Interpolation

Iterated Function Systems

Let $\mathsf{E} := (\mathsf{E}, d_{\mathsf{E}})$ denote a Banach space.

For $n \in \mathbb{N}$, let $\mathbb{N}_n := \{1, \ldots, n\}$.

For a map $f : \mathsf{E} \to \mathsf{E}$, the Lipschitz constant associated with f is

$$\mathrm{Lip}(f) := \sup_{x,y \in \mathsf{E}, x \neq y} \frac{\|f(x) - f(y)\|_{\mathsf{E}}}{\|x - y\|_{\mathsf{E}}}.$$

f is called Lipschitz if $\mathrm{Lip}(f) < +\infty$ and a contraction if $\mathrm{Lip}(f) < 1$.

Iterated function system (IFS): A collection of functions

$$\mathcal{F} := \mathcal{F}_n := \{f_i : \mathsf{E} \to \mathsf{E} : i \in \mathbb{N}_n\}.$$

The IFS $(\mathsf{E}; \mathcal{F})$ is called *contractive* if all $f \in \mathcal{F}$ are contractions.

Hyperspace of Nonempty Compact Subsets

Let E be a Banach space.

Hyperspace $\mathcal{H}(\mathsf{E})$ *of Compact Subsets of* E: collection of all nonempty compact subsets of E.

The Hausdorff-Pompeiu metric $d_{\mathcal{H}}$ on $\mathcal{H}(\mathsf{E})$ is defined by

$$d_{\mathcal{H}}(A, B) := \max\left\{\max_{a\in A}\min_{b\in B}\|a-b\|, \max_{b\in B}\min_{a\in A}\|b-a\|\right\}.$$

The completeness of E implies that $(\mathcal{H}(\mathsf{E}), d_{\mathcal{H}})$ is a complete metric space.

Contraction on $(\mathcal{H}(\mathsf{E}), d_{\mathcal{H}})$

Let $\mathcal{F} := \{f_i : \mathsf{E} \to \mathsf{E} : i \in \mathbb{N}_n\}$ be a contractive IFS.

The set-valued mapping $\mathcal{F} : \mathcal{H}(\mathsf{E}) \to \mathcal{H}(\mathsf{E})$

$$\mathcal{F}(A) := \bigcup_{i=1}^{n} f_i(A).$$

is contractive with Lipschitz constant $\operatorname{Lip}(\mathcal{F}) = \max_{i \in \mathbb{N}_n} \operatorname{Lip}(f_i) < 1$.

Banach Fixed Point Theorem $\Longrightarrow$ $\mathcal{F}$ has a unique fixed point F

$$F = \mathcal{F}(F) = \bigcup_{i=1}^{n} f_i(F) \quad \left(= \bigcup_{i_1=1}^{n} \cdots \bigcup_{i_k=1}^{n} f_{i_1} \circ \cdots \circ f_{i_k}(F) \right).$$

F is called the *attractor* or the *fractal* generated by the IFS $(\mathsf{E}; \mathcal{F})$.

$$F = \lim_{n \to \infty} \mathcal{F}^n(A), \quad A \in \mathcal{H}(\mathsf{E}) \text{ arbitrary}$$

Fractal Interpolation Functions (Barnsely 1986)

Let $\mathsf{E} := \mathbb{R}^2$ with the Euclidean norm. Given are

- $[a, b] \subset \mathbb{R}$ with $a < b$;
- $\mathsf{X} := [a, b] \times \mathbb{R} \subset \mathsf{E}$;
- $Y := \{(x_\nu, y_\nu) \in \mathsf{X} : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$;
- $s_i \in (-1, 1)$, $i \in \mathbb{N}_n$ (free parameters);
- $A := [a, b] \times [a, b]$.

For $i \in \mathbb{N}_n$, denote by A_i the parallelogram with vertices

$$(x_{i-1}, y_{i-1}),\ (x_i, y_i),\ (x_{i-1}, y_{i-1} + s_i(b-a)),\ (x_i, y_i + s_i(b-a)).$$

There exists a unique affine mapping $f_i : \mathsf{X} \to \mathsf{X}$ with $f_i(A) = A_i$.

$$f_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}, \quad i \in \mathbb{N}_N,$$

where

$$\begin{aligned} a_i &:= \frac{x_i - x_{i-1}}{b-a}, & c_i &:= \frac{y_i - y_{i-1} - s_i\,(y_n - y_0)}{b-a}, \\ \alpha_i &:= \frac{b x_{i-1} - a x_i}{b-a}, & \beta_i &:= \frac{b y_{i-1} - a y_i - s_i\,(b y_0 - a y_n)}{b-a}. \end{aligned}$$

There exists a norm - equivalent to the Euclidean norm - such that $\mathrm{Lip}(f_i) \le q < 1$. Thus, the IFS $(\mathsf{X}, \mathcal{F})$ with $\mathcal{F} := \{f_i : i \in \mathbb{N}_n\}$ possesses a unique fixed point $G \in \mathcal{H}(\mathsf{X})$.

G is the graph of a continuous function $\psi : [a,b] \to \mathbb{R}$ interpolating the set Y: $\psi(x_\nu) = y_\nu$, $\nu \in \{0\} \cup \mathbb{N}_n$.

Barnsley, M.F. *Fractal Functions and Interpolation*, Constr. Approx. (1986) 2, 303–329.

Example of Construction of FIF

Let $n := 3$, $\mathsf{X} := [0,1] \times \mathbb{R}$.

$Y := \{(0,0), (\frac{1}{2}, \frac{7}{10}), (\frac{7}{10}, -\frac{1}{10}), (1, \frac{3}{10})\}$.

Let $(s_1, s_2, s_3) := (\frac{3}{5}, -\frac{1}{2}, \frac{3}{4})$.

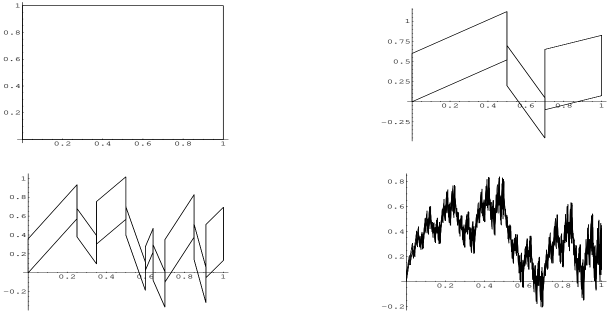

Figure: The geometric construction of a fractal interpolation function.

As the graph of ψ is, in general, a fractal set these functions were called by Barnsley *fractal interpolation functions.*

Since $G = \operatorname{graph}\psi$ and satisfies a fixed point equation, one has

$$\operatorname{graph}\psi = \bigcup_{i=1}^{n} f_i(\operatorname{graph}\psi)$$

and, therefore,

$$\begin{pmatrix} \bar{x} \\ \psi(\bar{x}) \end{pmatrix} \Bigg|_{\bar{x}\in[x_{i-1},x_i]} = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \begin{pmatrix} x \\ \psi(x) \end{pmatrix} \Bigg|_{x\in[a,b]} + \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}.$$

Let

$$l_i : [a,b] \to [x_{i-1}, x_i], \quad x \mapsto a_i x + \alpha_i,$$

and

$$q_i : [a,b] \to \mathbb{R}, \quad x \mapsto c_i x + \beta_i.$$

Note:

$$l_i(a) = x_{i-1} \quad \text{and} \quad l_i(b) = x_i, \quad i \in \mathbb{N}_n.$$

$\forall\, \bar{x} \in [x_{i-1}, x_i] \;\exists!\, x \in [a,b]\colon\; \bar{x} = l_i(x)$.

$$\begin{aligned}\psi(\bar{x}) &= c_i l_i^{-1}(\bar{x}) + s_i \psi(l_i^{-1}(\bar{x})) + \beta_i \\ &= (q_i \circ l_i^{-1})(\bar{x}) + s_i (\psi \circ l_i^{-1})(\bar{x}), \quad \forall\, \bar{x} \in [x_{i-1}, x_i].\end{aligned}$$

Functional Equations for ψ

$$\psi(x) = \sum_{i\in\mathbb{N}_n} (q_i \circ l_i^{-1})(x)\mathbb{1}_{[x_{i-1},x_i]}(x) + \sum_{i\in\mathbb{N}_n} s_i\,(\psi \circ l_i^{-1})(x)\mathbb{1}_{[x_{i-1},x_i]}(x),$$

respectively,

$$\psi(l_i(x)) = q_i(x) + s_i\,\psi(x), \quad \forall\, x \in [a,b],\ i \in \mathbb{N}_n.$$

More generally (for later),

$$\psi(l_i(x)) = q_i(x) + s_i(x)\,\psi(x), \quad \forall\, x \in [a,b],\ i \in \mathbb{N}_n.$$

The Read–Bajraktarević (RB) Operator

Define an operator $T : C[a,b] \to C[a,b]$ by

$$Tf := \sum_{i \in \mathbb{N}_n} (q_i \circ l_i^{-1}) \mathbb{1}_{[x_{i-1},x_i]} + \sum_{i=1}^{n} s_i (f \circ l_i^{-1}) \mathbb{1}_{[x_{i-1},x_i]}.$$

The operator T is a contraction on the Banach space $(C[a,b], \|\cdot\|_\infty)$ with Lipschitz constant $s := \max\{|s_i| : i \in \mathbb{N}_n\} < 1$.

The completeness of $(C[a,b], \|\cdot\|_\infty)$ implies the existence of a unique fixed point ψ which, by the Banach Fixed Point Theorem, can be obtained via a sequence $\{f_n : n \in \mathbb{N}\} \subset C[a,b]$ given by

$$f_n := T f_{n-1},$$

where $f_0 \in C[a,b]$ is arbitrary.

Relation between IFS and RB Operator

$$\begin{array}{ccc} \mathcal{H}(\mathsf{X}) & \xrightarrow{\ \mathcal{F}\ } & \mathcal{H}(\mathsf{X}) \\ \big\uparrow{\scriptstyle G} & & \big\uparrow{\scriptstyle G} \\ C[a,b] & \xrightarrow{\ T\ } & C[a,b] \end{array}$$

where G is the mapping

$$C[a,b] \ni g \mapsto G(g) = \{(x, g(x)) : x \in [a,b]\} \in \mathcal{H}(\mathsf{X}).$$

Fractal Interpolation Problem

Given a bounded subset X of a Banach space E and a Banach space F, construct a global function $\psi : \mathsf{X} = \coprod_{i=1}^{n} \mathsf{X}_i \to \mathsf{F}$ belonging to some prescribed function space $\mathscr{F} := \mathscr{F}(\mathsf{X}, \mathsf{F})$ satisfying n functional equations of the form

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad \text{on } \mathsf{X} \text{ and for } i \in \mathbb{N}_n,$$

where the functions l_i partition X into disjoint subsets $\mathsf{X}_i = l_i(\mathsf{X})$, $q_i \in \mathscr{F}$, and the functions s_i are chosen so that

$$s_i(x)\psi(x) \in \mathsf{F} \quad \text{and} \quad s_i\,\psi \in \mathscr{F}.$$

Global Fractal Interpolation - Bounded Solutions

Let X be a nonempty bounded subset of E.

$\{l_i\}_{i=1}^n$ of injective contractions $\mathsf{X} \to \mathsf{X}$ generating a partition of X:

$$\mathsf{X} = \coprod_{i=1}^{n} l_i(\mathsf{X}) =: \coprod_{i=1}^{n} \mathsf{X}_i.$$

Let $\mathcal{B}(\mathsf{X}, \mathsf{F}) := \{f : \mathsf{X} \to \mathsf{F} : f \text{ is bounded}\}$ denote the the Banach space of bounded functions equipped with $\|f\| := \sup\limits_{x \in \mathsf{X}} \|f(x)\|_{\mathsf{F}}$.

Define an RB operator $T : \mathcal{B}(\mathsf{X}, \mathsf{F}) \to \mathsf{F}^{\mathsf{X}}$

$$Tf(x) = (q_i \circ l_i^{-1})(x) + (s_i \circ l_i^{-1})(x) \cdot (f \circ l_i^{-1})(x),$$

for $x \in \mathsf{X}_i$ and $i \in \mathbb{N}_n$

Existence of Bounded Solution

Theorem. *The system of functional equations*

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad \text{on } \mathsf{X} \text{ and for } i \in \mathbb{N}_n,$$

has a unique bounded solution $\psi : \mathsf{X} \to \mathsf{F}$ *provided that*

(a) $\mathsf{X} = \coprod_{i=1}^{n} \mathsf{X}_i$,

(b) $q_i \in \mathcal{B}(\mathsf{X}, \mathsf{F})$, $s_i : \mathsf{X} \to \mathbb{R}$, *and*

(c) $s := \max_{i\in\mathbb{N}_n} \sup_{x\in\mathsf{X}} |s_i(x)| < 1$.

- Massopust, P.R. *Fractal Functions, Fractal Surfaces, and Wavelets*, 2nd ed., Academic Press: San Diego, USA, 2016.
- Serpa, C.; Buescu, J. J. Constructive solutions for systems of iterative functional equations. *Constr. Approx.*, **2017**, *45(2)*, 273–299.

Example

- $\mathsf{E} := \mathbb{R} =: \mathsf{F}$.
- $\mathsf{X} := [0, 1) \subset \mathsf{E}$.
- $l_i : [0, 1) \to [0, 1)$ with $l_1(x) := \frac{1}{3}x$ and $l_2(x) := \frac{2}{3}x + \frac{1}{3}$.
- Thus, $\mathsf{X}_1 = [0, \frac{1}{3})$ and $\mathsf{X}_2 = [\frac{1}{3}, 1)$. Clearly, $\mathsf{X} = \coprod_{i=1}^{2} \mathsf{X}_i$.
- $q_1(x) = -1$ and $q_2(x) = x$
- $s_1(x) = \frac{1}{2}\sin(x)$ and $s_2(x) := -\frac{2}{3}\cos(x)$.
- System of functional equations:

$$\psi(\tfrac{1}{3}x) = -1 + \tfrac{1}{2}\sin(x)\psi(x) \quad \text{and} \quad \psi(\tfrac{2}{3}x + \tfrac{1}{3}) = x - \tfrac{2}{3}\cos(x)\psi(x),$$

- Associated RB operator:

$$Tf(x) = \begin{cases} -1 + \frac{1}{2}\sin(3x)f(3x), & 0 \le x < \frac{1}{3}; \\ 3x - 1 - \frac{2}{3}\cos(\frac{1}{2}(3x-1))f(\frac{1}{2}(3x-1)), & \frac{1}{3} \le x < 1, \end{cases}$$

$s = \frac{2}{3} < 1 \Longrightarrow T$ is contractive.

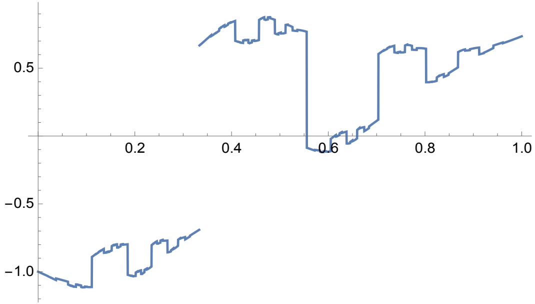

Figure: The solution/fixed point ψ.

Global Fractal Interpolation - L^p solutions

$\mathsf{X} \subset \mathsf{E} := \mathbb{R}^m$ and $\mathsf{Y} := \mathbb{R}^k$.

$\psi \in L^p(\mathsf{X}, \mathbb{R}^k) \quad \Longrightarrow \quad T : L^p(\mathsf{X}, \mathbb{R}^k) \to L^p(\mathsf{X}, \mathbb{R}^k)$

$q_i \in L^p(\mathsf{X}, \mathbb{R}^k)$ and $s_i \in L^\infty(\mathsf{X}, \mathbb{R}^k)$.

Theorem. *The system of functional equations*

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad \text{on } \mathsf{X} \subset \mathbb{R}^m \text{ and for } i \in \mathbb{N}_n,$$

has a unique solution $\psi \in L^p(\mathsf{X}, \mathbb{R}^k)$, $1 \le p < \infty$ *provided that*

$$\sum_{i=1}^n \lambda_i s_i^p < 1,$$

where $\lambda_i = \left\|(l_i^{-1})'\right\|_\infty$ *and* $s_i = \|s_i\|_\infty$.

Massopust, P.R. *Fractal Functions, Fractal Surfaces, and Wavelets*, 2nd ed., Academic Press: San Diego, USA, 2016.

Global Fractal Interpolation - Continuous Solutions

Theorem. *The system of functional equations*

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad \text{on } \mathsf{X} \subset \mathbb{R}^m \text{ and for } i \in \mathbb{N}_n,$$

has a unique continuous solution $\psi : \mathsf{X} \to \mathsf{F}$ *provided that*

1. *the functions* l_i, q_i, *and* s_i *are continuous,*
2. $\forall\, i, j \in \mathbb{N}_n$ *and* $\forall\, x_1, x_2 \in \mathsf{X}$*:*

$$\lim_{x \to x_1} f_j(x) = f_i(x_2) \implies \lim_{x \to x_1} q_j(x) + s_j(x)\psi(x) = q_i(x_2) + s_i(x_2)\psi(x_2).$$

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- Serpa, C.; Buescu, J. J. Constructive solutions for systems of iterative functional equations. *Constr. Approx.*, **2017**, *45(2)*, 273–299.

Example

- $\mathsf{X} := [0, 1]$.
- $q_1(x) := x$ and $q_2(x) := 1 - x$.
- $s_1(x) = \frac{1}{2}\sin(x)$, and $s_2(x) := -\frac{2}{3}\cos(x)$.
- Note that here we have $\mathsf{X}_1 \cap \mathsf{X}_2 = \{\frac{1}{3}\}$ and $l_1(1) = \frac{1}{3} = l_2(0)$
- $q_1(1) + s_1(1)\psi(1) = q_2(0) + s_2(0)\psi(0)$. The functional equations imply for $x \in \{0, 1\}$

  $$\psi(0) = q_1(0) + s_1(0)\psi(0) \quad \text{and} \quad \psi(1) = q_2(1) + s_2(1)\psi(0),$$

  which gives the values of ψ at the endpoints of X:

  $$\psi(0) = \frac{q_1(0)}{1 - s_1(0)} \quad \text{and} \quad \psi(1) = \frac{q_2(1)}{1 - s_2(1)}.$$

- There exists a *bounded* solution ψ (since $s = \frac{2}{3} < 1$).
- $\lim_{x \to 0-} q_2(x) + s_2(x)\psi(x) = q_1(1) + s_1(1)\psi(1)$.

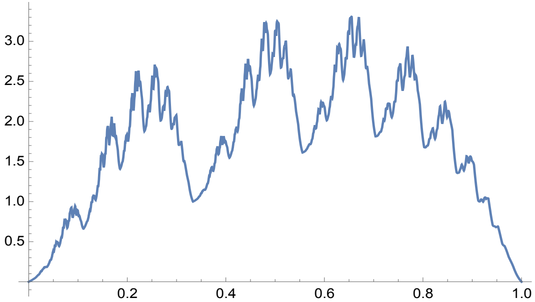

Figure: A continuous solution/fixed point ψ.

Local Fractal Interpolation

Let $\{\mathsf{X}_i : i \in \mathbb{N}_n\}$ be a family of nonempty subsets of a fixed nonempty bounded subset X of a normed space E.

Suppose $\{l_i\}_{i=1}^n$ is a collections of injective mappings from $\mathsf{X}_i \to \mathsf{X}$ generating a partition of X: $\mathsf{X} = \coprod_{i=1}^{n} l_i(\mathsf{X}_i)$.

Note that the l_i *need not* be contractive mappings here.

Local fractal interpolation looks for local solutions

$$\psi : \mathsf{X} = \bigcup_{i \in \mathbb{N}_n} l_i(\mathsf{X}_i) \to \mathsf{F}$$

of functional equations or for fixed points of RB operators of the form

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad x \in \mathsf{X}_i,\ i \in \mathbb{N}_n,$$

respectively,

$$Tf = (q_i \circ l_i^{-1}) + (s_i \circ l_i^{-1}) \cdot (f_i \circ l_i^{-1}), \quad x \in l_i(\mathsf{X}_i), \quad i \in \mathbb{N}_n,$$

where $f_i := f|_{\mathsf{X}_i}$, on appropriate function spaces.

Local Fractal Interpolation - Bounded Solutions.

- $s_i \in \mathcal{B}(\mathsf{X}_i, \mathbb{R})$ and
- $q_i \in \mathcal{B}(\mathsf{X}_i, \mathsf{F})$.

Theorem. *The system of functional equations*

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad x \in \mathsf{X}_i,\ i \in \mathbb{N}_n,$$

has a unique bounded solution $\psi : \mathsf{X} \to \mathsf{F}$, respectively, the RB operator has a unique bounded fixed point $\psi : \mathsf{X} \to \mathsf{F}$ provided that

1. $\mathsf{X} = \coprod_{i=1}^{n} \mathsf{X}_i$ *and*
2. $s := \max_{i \in \mathbb{N}_n} \sup_{x \in \mathsf{X}_i} |s_i(x)| < 1.$

Massopust, P.R. Local Fractal Functions in Besov and Triebel-Lizorkin Spaces. *J. Math. Anal. Appl.* **2016**, *436*, 393 – 407.

Local Fractal Interpolation - L^p Solutions

- $\mathsf{E} := \mathbb{R} =: \mathsf{F}$
- Partition of X: $\Delta := (0 =: x_0 < x_1 < \cdots < x_{n-1} < x_n := 1)$.
- $\{\mathsf{X}_i : i \in \mathbb{N}_n\}$ is a family of half-open intervals of $[0, 1]$.
- Affine mappings $l_i : \mathsf{X}_i \to [x_{i-1}, x_i)$ and $l_n : \mathsf{X}_n \to [x_{n-1}, x_n]$.

Theorem. *Assume that* $q_i \in L^p(\mathsf{X}_i, [0,1])$ *and* $s_i \in L^\infty(\mathsf{X}_i, \mathbb{R})$, $i \in \mathbb{N}_n$. *The system of functional equations*

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad x \in [0,1],\ i \in \mathbb{N}_n,$$

has a unique solution $\psi \in L^p[0,1]$, $1 \le p \le \infty$

$$\begin{cases} \left(\sum_{i=1}^{n} a_i \, \|s_i\|_{\infty,\mathsf{X}_i}^p\right)^{1/p}, & p \in [1,\infty); \\ \max_{i \in \mathbb{N}_n} \|s_i\|_{\infty,\mathsf{X}_i}, & p = \infty, \end{cases} < 1,$$

where a_i *Lipschitz constant of* $(l_i^{-1})'$ *and* $\|s_i\|_{\infty,\mathsf{X}_i} = \sup_{x \in \mathsf{X}_i} |s_i(x)|$.

Massopust, P.R. Local Fractal Functions in Besov and Triebel-Lizorkin Spaces. *J. Math. Anal. Appl.* **2016**, *436*, 393 – 407.

Local Fractal Interpolation - Continuous Solutions

Theorem. *The system of functional equations*

$$\psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad x \in [0,1],\ i \in \mathbb{N}_n,$$

has a unique continuous solution $\psi : \mathsf{X} \to \mathsf{F}$ *provided that*

1. *the functions* l_i, q_i, *and* s_i *are continuous,*
2. *and* $\forall i, j \in \mathbb{N}_n$, $i \neq j$, $\forall x_1 \in \mathsf{X}$, $\forall x_2 \in \mathsf{X}_i$*:*

$$\lim_{\substack{x \to x_1 \\ x \in \mathsf{X}_j}} f_j(x) = f_i(x_2)$$

$$\implies \lim_{\substack{x \to x_1 \\ x \in \mathsf{X}_j}} q_j(x) + s_j(x)\psi(x) = q_i(x_2) + s_i(x_2)\psi(x_2).$$

- Massopust, P.R. *Fractal Functions, Fractal Surfaces, and Wavelets*, 2nd ed., Academic Press: San Diego, USA, 2016.
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Non-Stationary Fractal Interpolation

- X is nonempty bounded subset of a normed space E
- Doubly-indexed family of injective contractions $\{l_{i_k,k} : i_k \in \mathbb{N}_{n_k},\, k \in \mathbb{N}\}$ from $\mathsf{X} \to \mathsf{X}$ generating a partition of X for each $k \in \mathbb{N}$.
- F Banach space
- $\{q_{i_k,k} : i_k \in \mathbb{N}_{n_k},\, k \in \mathbb{N}\} \subset \mathcal{B}(\mathsf{X}, \mathsf{F})$, and $\{s_{i_k,k} : i_k \in \mathbb{N}_{n_k},\, k \in \mathbb{N}\} \subset \mathcal{B}(\mathsf{X}, \mathbb{R})$ are such that

$$s := \sup_{k\in\mathbb{N}} \max_{i_k\in\mathbb{N}_k} \|s_{i_k,k}\|_\infty < 1.$$

- For each $k \in \mathbb{N}$, define RB operator $T_k : \mathcal{B}(\mathsf{X}, \mathsf{F}) \to \mathcal{B}(\mathsf{X}, \mathsf{F})$

$$(T_k f)(l_{i_k,k}(x)) := q_{i_k,k}(x) + s_{i_k,k}(x) \cdot f(x), \quad \forall x \in \mathsf{X}.$$

- T_k is a contraction on $\mathcal{B}(\mathsf{X}, \mathsf{F})$ with Lipschitz constant

$$\mathrm{Lip}(T_k) = \max_{i_k\in\mathbb{N}_k} \|s_{i_k,k}\|_\infty \le s < 1.$$

Invariant Set of a Sequence of Transformations

Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of transformations $T_k : \mathcal{B}(\mathsf{X}, \mathsf{F}) \to \mathcal{B}(\mathsf{X}, \mathsf{F})$.
A subset $\mathscr{I}$ of $\mathcal{B}(\mathsf{X}, \mathsf{F})$ is called an invariant set of $\{T_k\}_{k\in\mathbb{N}}$ if

$$\forall\, k \in \mathbb{N}\, \forall\, x \in \mathscr{I} : T_k(x) \in \mathscr{I}.$$

Proposition. *Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of transformations on $\mathcal{B}(\mathsf{X}, \mathsf{F})$. Suppose there exists a $g \in \mathcal{B}(\mathsf{X}, \mathsf{F})$ such that for all $f \in \mathcal{B}(\mathsf{X}, \mathsf{F})$*

$$\|T_k f - g\| \le \mu\, \|f - g\| + M,$$

for some $\mu \in [0, 1)$ and $M > 0$. Then the ball $B_r(g)$ of radius $r = M/(1-\mu)$ centered at g is an invariant set for $\{T_k\}_{k\in\mathbb{N}}$.

- Levin, D.; Dyn, N.; Viswanathan, P. Non-stationary versions of fixed-point theory, with applications to fractals and subdivision. *J. Fixed Point Theory Appl.* **2019**, *21*, 1–25.
- Massopust, P.R. Non-Stationary Fractal Interpolation. *Mathematics* **2019**, *7(8)*, 1 – 14.

Existence of Invariant Set

Proposition. *Let* $\{T_k\}_{k\in\mathbb{N}}$ *be a sequence of RB operators on* $(\mathcal{B}(\mathsf{X},\mathsf{F}),\|\cdot\|)$*. Suppose that the elements of* $\{q_{i_k,k} : i_k \in \mathbb{N}_{n_k},\, k\in\mathbb{N}\}$ *satisfy*

$$\sup_{k\in\mathbb{N}} \max_{i_k\in\mathbb{N}_k} \|q_{i_k,k}\| \le M,$$

for some $M > 0$*. Then the ball* $B_r(0)$ *of radius* $r = M/(1-s)$ *centered at* $0 \in \mathcal{B}(\mathsf{X},\mathsf{F})$ *is an invariant set for* $\{T_k\}_{k\in\mathbb{N}}$*.*

Massopust, P.R. Non-Stationary Fractal Interpolation. *Mathematics* **2019**, *7(8)*, 1 – 14.

Backward and Forward Trajectories

Suppose that $f \in \mathcal{B}(\mathsf{X}, \mathsf{F})$ and that $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of RB operators on $\mathcal{B}(\mathsf{X}, \mathsf{F})$. The sequences

$$\Phi_k(f) := T_k \circ T_{k-1} \circ \cdots \circ T_1(f)$$

and

$$\Psi_k(f) := T_1 \circ T_2 \circ \cdots \circ T_k(f)$$

are called the *forward*, respectively, *backward trajectory* of f.

Levin, D.; Dyn, N.; Viswanathan, P. Non-stationary versions of fixed-point theory, with applications to fractals and subdivision. *J. Fixed Point Theory Appl.* **2019**, *21*, 1–25.

A Convergence Result

Theorem. *Suppose* $\{T_k\}_{k\in\mathbb{N}}$ *is a sequence of RB operators on* $\mathcal{B}(\mathsf{X}, \mathsf{F})$*. Further suppose that*

1. *there exists a nonempty closed invariant set* $\mathscr{I} \subseteq \mathcal{B}(\mathsf{X}, \mathsf{F})$ *for* $\{T_k\}_{k\in\mathbb{N}}$*;*
2. *and*
$$\sum_{k=1}^{\infty} \prod_{j=1}^{k} \operatorname{Lip}(T_j) < \infty.$$

Then the backward trajectories $\Psi_k(f_0)$ *converge for any initial* $f_0 \in \mathscr{I}$ *to a unique function* $\psi \in \mathscr{I}$*.*

- Levin, D.; Dyn, N.; Viswanathan, P. Non-stationary versions of fixed-point theory, with applications to fractals and subdivision. *J. Fixed Point Theory Appl.* **2019**, *21*, 1–25.
- Massopust, P.R. Non-Stationary Fractal Interpolation. *Mathematics* **2019**, *7(8)*, 1 – 14.

Non-Stationary Fractal Functions

Theorem. *The backwards trajectories* $\{\Psi_k\}_{k\in\mathbb{N}}$ *converge for any initial* $f_0 \in \mathscr{I}$ *to a unique function* $\psi \in \mathscr{I}$*, where* $\mathscr{I}$ *is the closed ball in* $\mathcal{B}(\mathsf{X},\mathsf{F})$ *of radius* $M/(1-s)$ *centered at* 0*.*

The fixed point ψ generated by a sequence $\{T_k\}$ of non-stationary RB operators is termed a *non-stationary fractal function of class* $\mathcal{B}(X,Y)$*.*

Massopust, P.R. Non-Stationary Fractal Interpolation. *Mathematics* **2019**, *7(8)*, 1 – 14.

Non-stationary Fractal Interpolation

- Let $\mathsf{X} := [0,1]$ and $\mathsf{F} := \mathbb{R}$.
- For $k \in \mathbb{N}$ let $\{l_{i_k,k} : i_k \in \mathbb{N}_{n_k},\, k \in \mathbb{N}\}$ be family of injections from $[0,1] \to [0,1]$ generating a partition of $[0,1]$.
- Assume w.l.o.g. that $l_{1,k}(0) = 0$ and $l_{n_k,k}(1) = 1$ and define

$$x_{i_k-1,k} := l_{i_k,k}(0), \quad x_{i_k,k} := l_{i_k,k}(1), \quad i_k \in \mathbb{N}_{n_k}$$

  where $x_{0,k} := 0$ and $x_{n_k,k} := 1$. Further assume that

$$0 = x_{0,k} < \cdots < x_{i_k-1,k} < x_{i_k,k} < \cdots x_{n_k,k} = 1.$$

- Let $f \in \mathcal{C}[0,1]$ be arbitrary.
- Define a metric subspace of $\mathcal{C}[0,1]$ by

$$\mathcal{C}_*[0,1] := \{g \in \mathcal{C}[0,1] : g(0) = f(0) \wedge g(1) = f(1)\}.$$

- Furthermore, let $b \in \mathcal{C}_*[0,1]$ be the unique affine function whose graph connects the points $(0, f(0))$ and $(1, f(1))$:

$$b(x) = (f(1) - f(0))x + f(0).$$

- Let $\{\mathcal{P}_k\}_{k\in\mathbb{N}}$ be a family of sets of points in $[0,1]\times\mathbb{R}$ where

$$\mathcal{P}_k := \{(x_{j_k}, f(x_{j,k}) \in [0,1]\times\mathbb{R} : j = 0,1,\ldots,n\}.$$

- For $k\in\mathbb{N}$, define an RB operator $T_k : \mathcal{C}_*[0,1] \to \mathcal{C}_*[0,1]$ by

$$T_k g = f + \sum_{i_k=1}^{n_k} s_{i_k,k} \circ l_{i_k,k}^{-1} \cdot (g-b) \circ l_{i_k,k}^{-1}\, \chi_{l_{i_k,k}[0,1]},$$

where $\{s_{i_k,k}\}_{i_k=1}^{n_k} \subset \mathcal{C}[0,1]$ such that

$$\sup_{k\in\mathbb{N}} \max_{i_k\in\mathbb{N}_{i_k}} \|s_{i_k,k}\|_\infty < 1. \qquad (*)$$

- $T_k g$ is continuous at the points $x_{i_k,k}\in[0,1]$:

$$T_k g(x_{i_k,k}-) = T_k g(x_{i_k,k}+), \quad \forall\, i_k \in \{1,\ldots,n-1\}.$$

- $T_k g \in \mathcal{C}_*[0,1]$ and $T_k g$ interpolates $\mathcal{P}_k$ in the sense that

$$T_k g(x_{i_k,k}) = f(x_{i_k,k}), \quad \forall\, i_k \in \mathbb{N}_{n_k}.$$

Continuous Non-Stationary Fractal Function

Proposition. *A nonempty closed invariant set for* $\{T_k\}_{k\in\mathbb{N}}$ *is given by the closed ball in* $\mathcal{C}_*[0,1]$,

$$\mathscr{I} = \left\{ g \in \mathcal{C}_*[0,1] : \|g\| \leq \frac{\|f\| + s\|b\|}{1-s} \right\}.$$

Theorem. *Let* $\{T_k\}_{k\in\mathbb{N}}$ *be a sequence of RB operators each of whose elements acts on the complete metric space* $(\mathcal{C}_*[0,1], d)$ *where* $f \in \mathcal{C}_*[0,1]$ *is arbitrary and* b *is given as above. Furthermore, let the family of functions* $\{s_{i_k,k}\} \subset \mathcal{C}[0,1]$ *satisfy* $(*)$. *Then, for any* $f_0 \in \mathscr{I}$, *the backward trajectories* $\Psi_k(f_0)$ *converge to a function* $\psi \in \mathscr{I}$ *which interpolates* $\mathcal{P}_k$.

$\psi \in \mathcal{C}_*[0,1]$ is called a *continuous non-stationary fractal function.*

Massopust, P.R. Non-Stationary Fractal Interpolation. *Mathematics* **2019**, *7(8)*, 1 – 14.

Example

Consider the two RB operators $T_i : C_*[0,1] \to C_*[0,1]$, $i = 1, 2$, defined by

$$(T_1 f)(x) = \begin{cases} -\frac{1}{2}\, f(4x), & x \in [0, \frac{1}{4}), \\ -\frac{1}{2} + \frac{1}{2}\, f(4x-1), & x \in [\frac{1}{4}, \frac{1}{2}), \\ \frac{1}{2}\, f(4x-2), & x \in [\frac{1}{2}, \frac{3}{4}), \\ \frac{1}{2} + \frac{1}{2}\, f(4x-3), & x \in [\frac{3}{4}, 1], \end{cases}$$

and

$$(T_2 f)(x) := \begin{cases} \frac{3}{4} f(2x), & x \in [0, \frac{1}{2}), \\ \frac{3}{4} + \frac{1}{4} f(2x-1), & x \in [\frac{1}{2}, 1]. \end{cases}$$

The RB operators T_1 and T_2 generate *Kiesswetter's fractal function* respectively, a *Casino function.*

Alternating sequence $\{T_i\}_{i\in\mathbb{N}}$ of RB operators

$$T_k := \begin{cases} T_1, & 10(j-1) < k \leq 10j-5, \\ T_2, & 10j-5 < k \leq 10j, \end{cases} \qquad j \in \mathbb{N}.$$

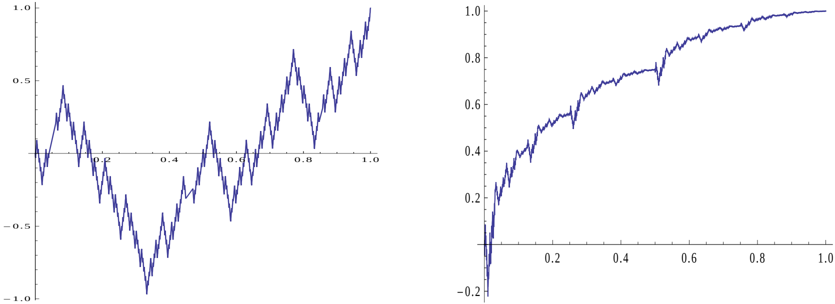


Figure: The hybrid Kiesswetter-Casino attractor.

Quaternionic Fractal Interpolation

Now, we extend fractal interpolation to a quaternionic setting.

As quaternions from a non-commutative division algebra over the reals, the non-commutativity generates more intricate and complex fractal patterns.

A Brief Introduction to Quaternions

Let $\{e_1, e_2, e_3\}$ be the canonical basis of the Euclidean vector space $\mathbb{R}^3$.

We call $\{e_1, e_2, e_3\}$ imaginary units and require that the following multiplication rules hold:

$$e_1^2 = e_2^2 = e_3^2 = -1,$$

$$e_1 e_2 = e_3 = -e_2 e_1, \quad e_2 e_3 = e_1 = -e_3 e_2, \quad e_3 e_1 = e_2 = -e_1 e_3.$$

A *real quaternion* q is then an expression of the form

$$q = a + \sum_{i=1}^{3} v_i e_i, \quad a, v_1, v_2, v_3 \in \mathbb{R}.$$

Addition and Multiplication

Let $q_1 = a + \sum_{i=1}^{3} v_i e_i$ and $q_2 = b + \sum_{i=1}^{3} w_i e_i$.

Addition: $q_1 + q_2 := (a + b) + \sum_{i=1}^{3} (v_i + w_i) e_i$.

Multiplication:

$$q_1 q_2 := (ab - v_1 w_1 - v_2 w_2 - v_3 w_3) + (a w_1 + b v_1 + v_2 w_3 - v_3 w_2) e_1 + \\ (a w_2 + b v_2 - v_1 w_3 + v_3 w_1) e_2 + (a w_3 + b v_3 + v - 1 w_2 - v_2 w_1) e_3.$$

Each quaternion $q = a + \sum_{i=1}^{3} v_i e_i$ may be decomposed as

$$q = \mathrm{Sc}(q) + \mathrm{Vec}(q),$$

where $\mathrm{Sc}(q) = a$ is the *scalar part* and $\mathrm{Vec}(q) = v = \sum_{i=1}^{3} v_i e_i$ is the *vector part* of q.

Conjugate and Inverse

The *conjugate* $\overline{q}$ of the real quaternion $q = a + v$ is $\overline{q} = a - v$.

Note that $q\overline{q} = \overline{q}q = a^2 + |v|^2 = a^2 + \sum_{i=1}^{3} v_i^2$.

Norm on $\mathbb{H}$: $|q| := \sqrt{q\overline{q}}$.

The inverse of a quaternion q is given by $q^{-1} = \frac{\overline{q}}{|q|^2}$.

$$\mathbb{H} := \mathbb{H}_{\mathbb{R}} := \left\{ a + \sum_{i=1}^{3} v_i e_i : a, v_1, v_2, v_3 \in \mathbb{R} \right\},$$

is a four-dimensional associative normed division algebra over $\mathbb{R}$.

Left Quaternionic Vector Spaces

A real vector space V is called a *left quaternionic vector space* if it is a left $\mathbb{H}$-module, i.e., if there exists a mapping $\mathbb{H} \times \mathsf{V} \to \mathsf{V}$, $(q, v) \mapsto qv$ which satisfies

1. $\forall v \in \mathsf{V} \, \forall q_1, q_2 \in \mathbb{H} : (q_1 + q_2)v = q_1 v + q_2 v.$
2. $\forall v_1, v_2 \in \mathsf{V} \, \forall q \in \mathbb{H} : q(v_1 + v_2) = qv_1 + qv_2.$
3. $\forall v \in \mathsf{V} \, \forall q_1, q_2 \in \mathbb{H} : q_1(q_2 v) = (q_1 q_2)v.$

A *two-sided quaternionic vector space* V is a left and right quaternionic vector space such that $\lambda v = v\lambda$, for all $\lambda \in \mathbb{R}$ and for all $v \in \mathsf{V}$.

Example of a two-sided quaternionic vector space is given by $\mathbb{H}$.

Quaternionic Normed Spaces

Let V be a left quaternionic vector space. A function $\|\cdot\| : \mathsf{V} \to \mathbb{R}_0^+$ is called a *norm* on V if

1. $\|v\| = 0$ iff $v = 0$.
2. $\|q\, v\| = |q|\, \|v\|$, for all $v \in \mathsf{V}$ and $q \in \mathbb{H}$.
3. $\|v + w\| \leq \|v\| + \|w\|$, for all $v, w \in \mathsf{V}$.

A left quaternionic vector space endowed with a norm will be called a *left quaternionic normed space.*

A left quaternionic normed space E is called *complete* if it is a complete metric space with respect to the metric $d(x, y) = \|x - y\|$ induced by the norm $\|\cdot\|$.

In this case, E is termed a *left quaternionic Banach space.*

Example

The space $\mathbb{H}^k$ consisting of k-tuples of quaternions is both a left and a right quaternionic vector space.

Represent elements $\xi \in \mathbb{H}^k$ as column vectors and define the quaternionic conjugate * of ξ by

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_k \end{pmatrix}^* := \begin{pmatrix} \overline{\xi_1} & \cdots & \overline{\xi_k} \end{pmatrix}, \qquad \xi_j \in \mathbb{H}.$$

$\mathbb{H}^k$ endowed with the norm $\|\xi\|_k := \sqrt{\xi^* \xi} = \left(\sum_{j=1}^{k} |\xi_j|^2 \right)^{1/2}$, becomes a two-sided quaternionic Banach space as $\lambda v = v\lambda$, $\forall\, \lambda \in \mathbb{R}$, $\forall\, v \in \mathbb{H}^k$.

$\mathbb{H}^k$ becomes a topological and a complete metric space under $\|\cdot\|_k$.

Left Linear Mappings

Let V_1 and V_2 be left quaternionic vector spaces. A mapping $f : \mathsf{V}_1 \to \mathsf{V}_2$ is called *left linear* if

$$f(q\,v + w) = qf(v) + f(w), \quad \forall v, w \in \mathsf{V}, \forall q \in \mathbb{H}.$$

A left linear mapping is termed *bounded* if

$$\|f\| := \sup_{x,y \in \mathsf{V}_1, x \neq y} \frac{\|f(x) - f(y)\|_{\mathsf{V}_2}}{\|x - y\|_{\mathsf{V}_1}} < \infty.$$

Quaternionic Fractal Interpolation

$\mathsf{E} := \mathbb{H} =: \mathsf{F}$.

Let $\mathsf{X} := \left\{ q = (q_0, q_1, q_2, q_3) \in \mathbb{H} : \max\limits_{i=0,1,2,3} |q_i| \leq 1 \right\} \cong [-1,1]^4$

A function $f : \mathsf{X} \to \mathbb{H}$ is called *bounded* if there exists a real number $M > 0$ such that $\|f\| \leq M$.

$$\mathcal{B}(\mathsf{X}, \mathbb{H}) := \{ f : \mathsf{X} \to \mathbb{H} : f \text{ is bounded} \}.$$

$\mathcal{B}(\mathsf{X}, \mathbb{H})$ becomes a left quaternionic vector space under

$$(f+g)(x) := f(x)+g(x) \quad \text{and} \quad (\lambda \cdot f)(x) := \lambda \cdot f(x), \quad \forall\, x \in \mathsf{X}\; \forall\, \lambda \in \mathbb{H}.$$

Setting for each $f \in \mathcal{B}(\mathsf{X}, \mathbb{H})$

$$\|f\| := \sup_{x \in \mathsf{X}} \|f(x)\|,$$

then $\mathcal{B}(\mathsf{X}, \mathbb{H})$ becomes a left Banach space.

Divide X into $n := 2^4$ congruent four-dimensional subcubes X_i each similar to X and such that $\{\mathsf{X}_i\}_{i=1}^n$ forms a partition of X.

Consider the RB operator $T : \mathcal{B}(\mathsf{X}, \mathbb{H}) \to \mathcal{B}(\mathsf{X}, \mathbb{H})$ given by

$$Tf(l_i(x)) := q_i(x) + s_i(x) f(x), \quad x \in \mathsf{X}, \ \ i \in \mathbb{N}_n,$$

where $q_i, s_i : \mathsf{X} \to \mathbb{H}$ are bounded functions.

Let $\mathcal{F} := \{f_1, \ldots, f_n\}$. Write

$$f_{i_m i_{m-1} \cdots i_1} := f_{i_m} \circ f_{i_{m-1}} \circ f_{i_1},$$

where each $i_j \in \mathbb{N}_n$.

For each $m \in \mathbb{N}$:

$$T^m f(l_{i_m i_{m-1} \cdots i_1}(x)) = \sum_{k=1}^{m} \prod_{j=1}^{k-1} s_{i_j}(x) q_k(x) + \prod_{k=1}^{m} s_{i_k}(x) f(x).$$

Bounded Quaternionic Fractal Function

Theorem. *For the above setting, the RB operator T has a unique fixed point $\psi \in \mathcal{B}(\mathsf{X}, \mathbb{H})$, i.e.,*

$$T\psi = \psi \quad \Longleftrightarrow \quad \psi(l_i(x)) = q_i(x) + s_i(x)\psi(x), \quad x \in \mathsf{X}, \;\; i \in \mathbb{N}_n,$$

provided that

$$\max_{i \in \mathbb{N}_n} \sup_{x \in \mathsf{X}_i} |s_i(x)| < 1.$$

The fixed point ψ is called a *bounded quaternionic fractal function.*

Massopust, P.R. Fractal interpolation: From global to local, to non-stationary and quaternionic, in *Frontiers of Fractal Analysis: Recent Advances and Challenges*, S. Banerjee & A. Gowrisankar (eds.), CRC Press, Boca Raton, 2022, 24 – 48.

Example

- $\mathsf{X} := \{q \in \mathbb{H} : \operatorname{Sc} q \in [0,1) \wedge \operatorname{Vec} q = 0\}$. Note that $\mathsf{X} \cong [0,1) \subset \mathbb{R}$.
- Define injections $l_i : \mathsf{X} \to \mathsf{X}$ as follows:

$$l_1(x) := \tfrac{1}{2}x \quad \text{and} \quad l_2(x) := \tfrac{1}{2}(x+1).$$

- Let $q_1 := e_0 + 2e_1 - e_3 + 3e_4$ and $q_2 := -e_0 - 2e_1 + 2e_3 + e_4$.
- Set $q_1(x) := (1 - q_1)x$ and $q_2(x) := q_2 x^2$. $(q_1, q_2 \in \mathcal{B}(\mathsf{X}, \mathbb{H}))$
- Define an RB operator T by

$$Tf(x) := \begin{cases} 2(1-q_1)x + s_1 f(2x), & x \in [0, \frac{1}{2}), \\ q_2(2x-1)^2 + s_2 f(2x-1), & x \in [\frac{1}{2}, 1), \end{cases}$$

- $s_1 := \frac{1}{10}e_0 + \frac{1}{2}e_1 - \frac{1}{5}e_2 - \frac{1}{10}e_3$ and $s_2 := -\frac{1}{5}e_0 + \frac{1}{5}e_1 - \frac{3}{5}e_2 + \frac{1}{10}e_3$.
- $|s_1| = \frac{1}{10}\sqrt{31}$ and $|s_2| = \frac{1}{10}\sqrt{45}$. Thus, $\max\{s_1, s_2\} < 1$.

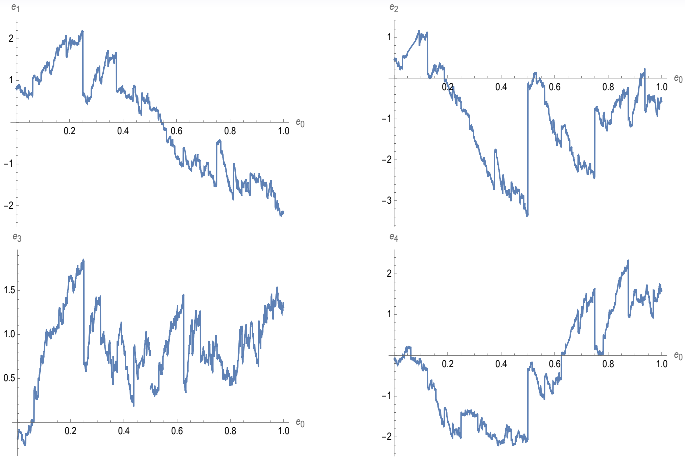
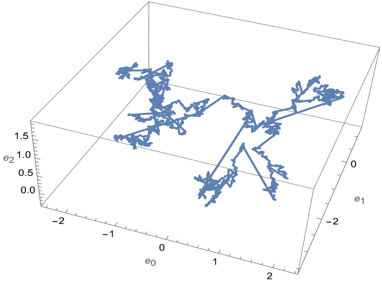


Figure: The projections of ψ onto the (e_0, e_i)-planes.

As ψ can be written as $\psi = \sum_{i=0}^{3} \psi_i e_i$, the parametric plots (ψ_0, ψ_1, ψ_2) and (ψ_0, ψ_2, ψ_4) are displayed below.



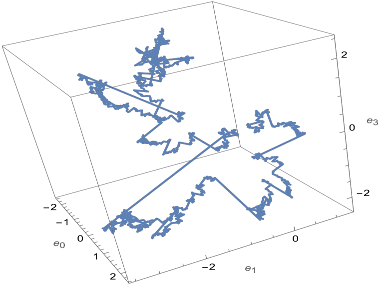


Figure: Some parametric plots of the components of ψ.

Mulţumesc Mult!